ESSENTIALS OF TRIGONOMETRY

WELLS

W. H. Mezger
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THE

ESSENTIALS

OF

PLANE AND SPHERICAL TRIGONOMETRY.

BY

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PREFACE.

This volume contains only those portions of Plane and Spherical Trigonometry which are essential in the practical applications of the subject to problems in surveying, geodesy, and navigation.

The attention of teachers and others is invited to the following, which may be regarded as constituting the salient features of the work:

1. The proofs of the functions of 0°, 90°, 180°, and 270°; Arts. 38 to 41.

2. The figures, and the tabular arrangement of the work, in the discussions of Arts. 42 and 44.

3. The solution of problems in Art. 56 by the construction of a diagram, in a manner analogous to that of Art. 15.

4. The proofs of the fundamental formulæ for any angle; Arts. 57 to 59.

5. The discussion of the line values of the trigonometric functions, and their application in tracing the changes in the functions as the angle increases from 0° to 360°; Arts. 62 to 64.

6. The proofs of the formulæ for the sines and cosines of $x + y$ and $x - y$ for any values of x and y; Arts. 68 to 70.

7. The discussion of the *ambiguous case* in the solution of plane oblique triangles; Arts. 124 to 126.

8. The *geometrical* proofs (Art. 138) of the propositions that in any spherical right triangle:

I. If the sides including the right angle are in the same quadrant, the hypotenuse is $< 90°$; if they are in different quadrants, the hypotenuse is $> 90°$.

II. An angle is in the same quadrant as its opposite side.

9. The discussion of the properties of spherical right triangles *before* those of spherical oblique triangles; see Chapters XI. and XII.

10. The reduction of the number of cases in the complete demonstration of the fundamental formulæ for spherical right triangles, to three, by application of the results proved geometrically in Art. 138; see Art. 143.

11. The discussions of the *ambiguous cases* in the solution of spherical oblique triangles (Arts. 171 and 172); especially the rules given on pages 130 and 132 for determining the number of solutions.

At the end of the book will be found a collection of formulæ in form for convenient reference.

Teachers who desire a briefer course are recommended to omit Chapter IV., which may be done without interrupting the logical completeness of the rest of the work. Chapter VI. may also be omitted by those who have taken the subject of Logarithms in their course in Algebra. The course might be still further abridged, if desired, by the omission of the exercises at the end of Chapter V.

WEBSTER WELLS.

BOSTON, 1887.

CONTENTS.

PART I. PLANE TRIGONOMETRY.

Part I.

PLANE TRIGONOMETRY.

I. DEFINITIONS; MEASUREMENT OF ANGLES.

1. *Trigonometry* is that branch of mathematics in which algebraic processes are used to treat of the properties and measurement of geometrical figures.

In *Plane Trigonometry* we consider *plane* figures only.

2. An angle is measured by finding its ratio to another angle adopted arbitrarily as the unit of measurement.

3. The usual unit of measurement for angles is the *degree*, or an angle equal to the ninetieth part of a right angle.

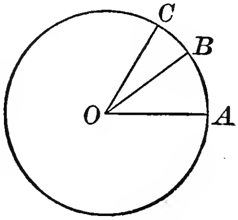
To express fractional parts of the unit, the degree is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

Degrees, minutes, and seconds are denoted by the symbols $^\circ$, $'$, $''$, respectively; thus, $43^\circ\,22'\,37''$ denotes an angle of 43 degrees, 22 minutes, and 37 seconds.

CIRCULAR MEASURE OF AN ANGLE.

4. Another method of measuring angles, and one of great importance, is known as the *circular method*, in which the unit of measurement is *the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.*

5. Let AOB be any angle, and AOC the unit of circular measure.



By Geometry, we have

$$\frac{\text{angle } AOB}{\text{angle } AOC} = \frac{\text{arc } AB}{\text{arc } AC}.$$

Whence, circular measure $AOB = \dfrac{\text{arc } AB}{OA}$ (Art. 4).

That is, *the circular measure of an angle is equal to the ratio of its subtending arc to the radius of the circle.*

For example, the circular measure of a right angle is equal to the ratio of one-fourth the circumference to the radius.

But the circumference of a circle is equal to the radius multiplied by 2π, where $\pi = 3.14159265 \ldots$

Hence, if R denotes the radius, we have

$$\text{circular measure of } 90° = \frac{\frac{1}{4} \text{ of } 2\pi R}{R} = \frac{\pi}{2}.$$

6. Since the circular measure of $90°$ is $\frac{\pi}{2}$, the circular measure of $180°$ is π; of $60°$, $\frac{\pi}{3}$; of $45°$, $\frac{\pi}{4}$; of $30°$, $\frac{\pi}{6}$; etc.

That is, *an angle expressed in degrees may be reduced to circular measure by finding its ratio to* $180°$, *and multiplying the result by* π.

Thus, since $115°$ is $\frac{23}{36}$ of $180°$, the circular measure of $115°$ is $\frac{23\pi}{36}$.

7. Conversely, *any angle expressed in circular measure may be reduced to degrees by multiplying by* 180° *and dividing by* π; or, more briefly, *by substituting* 180° *for* π.

Thus, $\frac{7\pi}{15} = \frac{7}{15}$ of $180° = 84°$.

8. In the circular method such expressions may occur as "the angle $\frac{2}{3}$," "the angle 1," etc.

These refer to the unit of circular measure; that is, the angle $\frac{2}{3}$ means an angle whose subtending arc is two-thirds of the radius.

The angle 1, that is, the angle whose subtending arc is equal to the radius, or the unit of circular measure, if reduced to degrees by the rule of Art. 7, gives

$$\frac{180°}{\pi} = \frac{180°}{3.14159265\ldots} = 57.2957795°\ldots$$

The rule of Art. 7 may then be modified as follows:

Any angle expressed in circular measure may be reduced to degrees by multiplying by 57.2957795°...

Thus, the angle $\frac{2}{3} = \frac{2}{3} \times 57.2957795°\ldots$

$$= 38.1971863°\ldots = 38°\,11'\,49.87068''\ldots$$

EXAMPLES.

9. Express the following in circular measure:

1. 135°. **3.** 11° 15′. **5.** 29° 15′. **7.** $128°\,34\frac{2}{7}'$.

2. 198°. **4.** 37° 30′. **6.** 174° 22′ 30″. **8.** 92° 48′ 45″.

Express the following in degree measure:

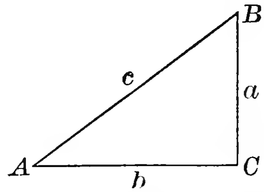
9. $\frac{1}{2}$ **11.** $\frac{37\pi}{30}$. **13.** $\frac{3}{4}$. **15.** $\frac{2\pi - 1}{3}$.

10. $\frac{3\pi}{5}$. **12.** $\frac{5\pi}{4}$. **14.** 2. **16.** $\frac{\pi - 1}{4}$.

II. THE TRIGONOMETRIC FUNCTIONS.

FUNCTIONS OF ACUTE ANGLES.

10. Let BAC be any acute angle.



From any point in either side, as B, draw BC perpendicular to the other side, forming a right triangle ABC.

We then have the following definitions, applicable to either of the acute angles A and B:

In any right triangle,

The SINE *of either acute angle is the ratio of the opposite side to the hypotenuse.*

The COSINE *is the ratio of the adjacent side to the hypotenuse.*

The TANGENT *is the ratio of the opposite side to the adjacent side.*

The COTANGENT *is the ratio of the adjacent side to the opposite side.*

The SECANT *is the ratio of the hypotenuse to the adjacent side.*

The COSECANT *is the ratio of the hypotenuse to the opposite side.*

That is, denoting the sides BC, CA, and AB by a, b, and c, and employing the usual abbreviations,

$$\left.\begin{array}{lll} \sin A = \frac{a}{c}, & \tan A = \frac{a}{b}, & \sec A = \frac{c}{b}, \\ \cos A = \frac{b}{c}, & \cot A = \frac{b}{a}, & \csc A = \frac{c}{a}. \end{array}\right\} \qquad (1)$$

And in like manner,

$$\left.\begin{array}{lll}\sin B=\dfrac{b}{c}, & \tan B=\dfrac{b}{a}, & \sec B=\dfrac{c}{a}, \\[2ex] \cos B=\dfrac{a}{c}, & \cot B=\dfrac{a}{b}, & \csc B=\dfrac{c}{b}.\end{array}\right\} \qquad (2)$$

11. The following definitions are also used:

The versed sine of an angle is equal to unity minus the cosine of the angle.

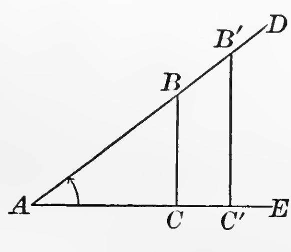
The coversed sine is equal to unity minus the sine.

That is, $\text{vers}\, A=1-\dfrac{b}{c}, \quad \text{covers}\, A=1-\dfrac{a}{c},$

$$\text{vers}\, B=1-\frac{a}{c}, \quad \text{covers}\, B=1-\frac{b}{c}.$$

12. The eight ratios defined in Arts. 10 and 11 are called *Trigonometric Functions*, or *Trigonometric Ratios*, of the angle.

It is important to observe that their values depend solely on the magnitude of the angle, and are entirely independent of the lengths of the sides of the right triangle which contains it.



Thus, let B and B' be any two points in the side AD of the angle DAE, and draw BC and $B'C'$ perpendicular to AE.

Then by Art. 10, we have

$$\sin A=\frac{BC}{AB}, \text{ and } \sin A=\frac{B'C'}{AB'}.$$

But the right triangles ABC and $AB'C'$ are similar, since they have the angle A common; and therefore, by Geometry,

$$\frac{BC}{AB}=\frac{B'C'}{AB'}.$$

Thus the two values obtained for sin A are seen to be equal.

13. We obtain from (1) and (2), Art. 10,

$$a=c\sin A, \qquad b=c\sin B,$$
$$b=c\cos A, \qquad a=c\cos B.$$

That is, *in any right triangle, either side about the right angle is equal to the hypotenuse multiplied by the sine of the opposite angle, or by the cosine of the adjacent angle.*

14. We have from Arts. 10 and 11,

$$\sin A=\frac{a}{c}=\cos B, \qquad \sin B=\frac{b}{c}=\cos A,$$

$$\tan A=\frac{a}{b}=\cot B, \qquad \tan B=\frac{b}{a}=\cot A,$$

$$\sec A=\frac{c}{b}=\csc B, \qquad \sec B=\frac{c}{a}=\csc A,$$

$$\operatorname{vers} A=1-\frac{b}{c}=\operatorname{covers} B, \qquad \operatorname{vers} B=1-\frac{a}{c}=\operatorname{covers} A.$$

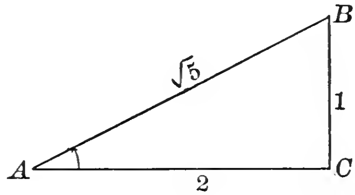
Since the angles A and B are complements of each other, the above results may be expressed as follows:

The sine, tangent, secant, and versed sine of an acute angle are respectively the cosine, cotangent, cosecant, and coversed sine of the complement of the angle.

It is from this circumstance that the names *co*-sine, *co*-tangent, etc., were derived.

15. The Pythagorean Theorem affords a simple method for finding the values of the remaining seven functions of an acute angle, when the value of any one is given.

1. Given $\cot A = 2$; required the values of the remaining functions of A.



The equation may be written $\cot A = \frac{2}{1}$.

Then since the cotangent is the ratio of the adjacent side to the opposite side, we may regard our value as having been taken from a right triangle ABC, having its side AC adjacent to the angle A equal to 2 units, and its side BC opposite to A equal to 1 unit.

But by Geometry, we have

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{4+1} = \sqrt{5}.$$

Whence by definition,

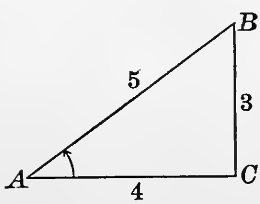
$$\sin A = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \qquad \sec A = \frac{\sqrt{5}}{2},$$

$$\cos A = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}, \qquad \csc A = \sqrt{5},$$

$$\tan A = \frac{1}{2}, \qquad \text{vers } A = 1 - \frac{2\sqrt{5}}{5},$$

$$\text{covers } A = 1 - \frac{\sqrt{5}}{5}.$$

2. Given covers $A = \frac{2}{5}$; required the values of the remaining functions of A.



Since covers $A = 1 - \sin A$, we have $\sin A = 1 - \frac{2}{5} = \frac{3}{5}.$

We then take the opposite side BC equal to 3 units, and the hypotenuse AB equal to 5 units.

Then, $$AC = \sqrt{\overline{AB}^2 - \overline{BC}^2} = \sqrt{25 - 9} = 4.$$

Whence by definition,

$$\cos A = \frac{4}{5}, \qquad \cot A = \frac{4}{3}, \qquad \csc A = \frac{5}{3},$$

$$\tan A = \frac{3}{4}, \qquad \sec A = \frac{5}{4}, \qquad \text{vers } A = \frac{1}{5}.$$

EXAMPLES.

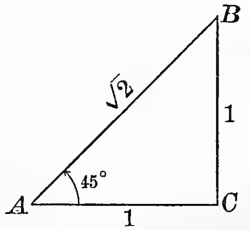
In each case find the values of the remaining functions:

3. $\tan A = \frac{2}{3}.$ **6.** $\text{vers } A = \frac{1}{4}.$ **9.** $\cot A = x.$

4. $\text{covers } A = \frac{3}{5}.$ **7.** $\sin A = \frac{x}{y}.$ **10.** $\cos A = \frac{8}{17}.$

5. $\csc A = 4.$ **8.** $\sec A = \frac{13}{5}.$ **11.** $\sec A = \frac{\sqrt{a^2 + b^2}}{b}.$

16. *To find the values of the sine, cosine, tangent, cotangent, secant, and cosecant of* 45°.



Let ABC be an isosceles right triangle, having each of the sides AC and BC equal to 1.

Then, $$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{1 + 1} = \sqrt{2}.$$

Also, the angles A and B are equal; and since their sum is a right angle, each is equal to 45°.

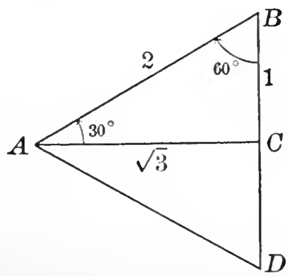
Then by definition,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \qquad \tan 45° = 1. \qquad \sec 45° = \sqrt{2}.$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \qquad \cot 45° = 1. \qquad \csc 45° = \sqrt{2}.$$

The second line might have been derived from the first by aid of Art. 14, since 45° is complement of itself.

17. *To find the values of the sine, cosine, tangent, cotangent, secant, and cosecant of* 30° *and* 60°.



Let ABD be an equilateral triangle, having each side equal to 2. Draw AC perpendicular to BD; then by Geometry,

$$BC = \tfrac{1}{2} BD = 1, \text{ and } \angle BAC = \tfrac{1}{2} \angle BAD = 30°.$$

Again, $AC = \sqrt{\overline{AB}^2 - \overline{BC}^2} = \sqrt{4 - 1} = \sqrt{3}.$

Then from the triangle ABC, by definition,

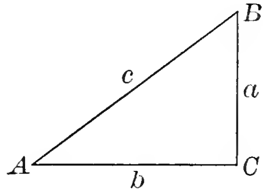
$$\sin 30° = \frac{1}{2} \qquad = \cos 60°. \qquad \cos 30° = \frac{\sqrt{3}}{2} = \sin 60°.$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \cot 60°. \qquad \cot 30° = \sqrt{3} = \tan 60°.$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \csc 60°. \qquad \csc 30° = 2 \quad = \sec 60°.$$

Or the functions of 60° may be derived from those of 30° by aid of Art. 14, since 60° is the complement of 30°.

FUNDAMENTAL THEOREMS.



18. We obtain from the definitions of Art. 10,

$$\sin A \csc A = \frac{a}{c} \times \frac{c}{a} = 1.$$

$$\cos A \sec A = \frac{b}{c} \times \frac{c}{b} = 1.$$

$$\tan A \cot A = \frac{a}{b} \times \frac{b}{a} = 1.$$

These results may be written

$$\left.\begin{array}{lll} \sin A = \dfrac{1}{\csc A}, & \tan A = \dfrac{1}{\cot A}, & \sec A = \dfrac{1}{\cos A}, \\ \cos A = \dfrac{1}{\sec A}, & \cot A = \dfrac{1}{\tan A}, & \csc A = \dfrac{1}{\sin A}. \end{array}\right\} \quad (3)$$

That is, *the sine of an acute angle is the reciprocal of the cosecant, the tangent is the reciprocal of the cotangent, and the secant is the reciprocal of the cosine.*

19. *To prove the formula*

$$\sin^2 A + \cos^2 A = 1.$$

Note. $\text{Sin}^2 A$ signifies $(\sin A)^2$; that is, the square of the sine of A.

By Geometry, $a^2 + b^2 = c^2$.

Dividing by c^2, $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$

Whence by definition,

$$(\sin A)^2 + (\cos A)^2 = 1;$$

or,
$$\sin^2 A + \cos^2 A = 1. \quad (4)$$

The result may be written in the forms

$$\sin A = \sqrt{1 - \cos^2 A}, \text{ and } \cos A = \sqrt{1 - \sin^2 A}.$$

20. *To prove the formulæ*

$$\tan A = \frac{\sin A}{\cos A}, \textit{ and } \cot A = \frac{\cos A}{\sin A}.$$

By definition,
$$\tan A = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin A}{\cos A}, \quad (5)$$

and
$$\cot A = \frac{b}{a} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\cos A}{\sin A}. \quad (6)$$

21. *To prove the formulæ*

$$\sec^2 A = 1 + \tan^2 A, \textit{ and } \csc^2 A = 1 + \cot^2 A.$$

By Geometry,
$$c^2 = a^2 + b^2.$$

Dividing by b^2,
$$\frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}.$$

That is,
$$\sec^2 A = 1 + \tan^2 A. \quad (7)$$

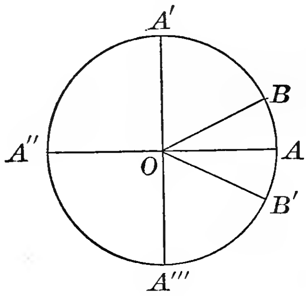
Dividing by a^2,
$$\frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}.$$

That is,
$$\csc^2 A = 1 + \cot^2 A. \quad (8)$$

III. APPLICATION OF ALGEBRAIC SIGNS.

TRIGONOMETRIC FUNCTIONS OF ANGLES IN GENERAL.

22. In Geometry we are, as a rule, concerned with angles less than two right angles; but in Trigonometry it is convenient to regard them as unrestricted in magnitude.



In the circle AA'', let AA'' and $A'A'''$ be a pair of perpendicular diameters.

Suppose a radius OB to start from the position OA, and revolve about the point O as a pivot in the direction of OA'.

When it coincides with OA', it has described an angular magnitude of $90°$; when it coincides with OA'', of $180°$; with OA''', of $270°$; with OA, its starting-point, of $360°$; with OA' again, of $450°$; etc. We thus see that a significance may be attached to any positive angular magnitude.

23. The interpretation of an angle as the measure of the amount of rotation of a moving radius, enables us to distinguish between positive and negative angles.

Thus, if a positive angle is taken to indicate revolution from the position OA in the direction of OA', a negative angle would naturally be taken as signifying revolution from the position OA' in the *opposite* direction, or towards OA'''.

That is, if the angles AOB and AOB' are each equal to $30°$ in absolute value, we should say that $AOB = +30°$, and $AOB' = -30°$.

We may thus conceive of *negative* angles of any magnitude whatever. It is immaterial which direction we consider the positive direction of rotation; but having adopted a certain direction as positive, our subsequent operations must be in accordance.

24. The fixed line OA, from which the rotation is supposed to commence, is called the *initial line*, and the final position of the rotating radius is called the *terminal line*.

Either side of an angle may be taken as the initial line, the other being then the terminal line; thus, in the angle AOB, we may consider OA the initial line and OB the terminal line, in which case the angle is positive; or we may consider OB the initial line and OA the terminal line, in which case the angle is negative.

25. In designating an angle, we shall always write first the letter at the extremity of the initial line; thus, in designating the angle AOB, if we regard OA as the initial line, we should call it AOB, and if we regard OB as the initial line, we should call it BOA.

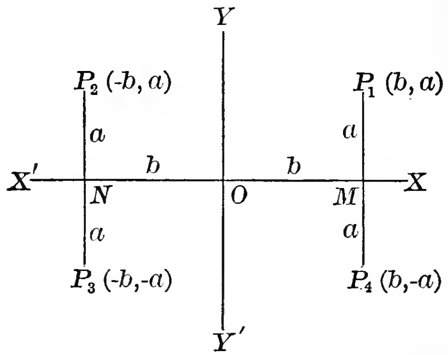
26. There are always two angles in absolute value less than $360°$, one positive and the other negative, formed by a given initial and terminal line.

Thus, there are formed by OA and OB' the positive angle AOB' greater than $270°$, and the negative angle AOB' less than $90°$.

We shall distinguish between such angles by referring to them as "the positive angle AOB'," and "the negative angle AOB'," respectively.

27. It is evident that the terminal lines of two angles which differ by a multiple of $360°$ are coincident; thus, the angles $30°$, $390°$, $-330°$, etc., have the same terminal line.

RECTANGULAR CO-ORDINATES.



28. Let XX' and YY' be a pair of straight lines at right angles to each other; let P_1 be any point in their plane, and draw P_1M perpendicular to XX'.

The distances OM and P_1M are called the *rectangular co-ordinates* of P_1 with reference to the lines XX' and YY'.

OM is called the *abscissa*, and P_1M the *ordinate;* the lines XX' and YY' are called the *axes of* X *and* Y, respectively, and their intersection O is called the *origin*.

29. If in the above figure, $OM = ON = b$, and the perpendiculars $P_1M = P_2N = P_3N = P_4M = a$, the points P_1, P_2, P_3, and P_4 will have the same co-ordinates.

To avoid this ambiguity, the following conventions have been adopted:

Abscissas measured to the *right* of O are considered *positive*, and to the *left*, *negative*.

Ordinates measured *above* the line XX' are considered *positive*, and *below*, *negative*.

Then the co-ordinates of the four points will be:

Point.	*Abscissa.*	*Ordinate.*
P_1	b	a
P_2	$-b$	a
P_3	$-b$	$-a$
P_4	b	$-a$

Note. In the figures of this chapter, the small letters denote the *lengths* of the lines, without regard to their algebraic sign.

30. It is customary to denote the abscissa and ordinate of a point by the letters x and y, respectively. Thus the fact that the abscissa of a point is equal to b and its ordinate to a, is expressed by the saying that for the point in question $x = b$ and $y = a$.

The same fact may be stated more concisely by referring to the point as "the point (b, a)", where the first quantity in the parenthesis is understood to be the abscissa, and the second the ordinate.

31. If a point lies upon the axis of X, its ordinate is zero; and the same is true of the abscissa of a point upon the axis of Y.

GENERAL DEFINITIONS OF THE FUNCTIONS.

32. We will now give general definitions for the trigonometric functions, applicable to any angle whatever.

Take the initial line as the positive direction of the axis of X, the vertex being the origin.

Take any point in the terminal line, and construct its rectangular co-ordinates by dropping a perpendicular to the initial line, produced if necessary.

Then, designating the distance of the assumed point from the origin as the "distance" of the point,

The SINE *is the ratio of the* ORDINATE *to the* DISTANCE.
The COSINE *is the ratio of the* ABSCISSA *to the* DISTANCE.
The TANGENT *is the ratio of the* ORDINATE *to the* ABSCISSA.
The COTANGENT *is the ratio of the* ABSCISSA *to the* ORDINATE.
The SECANT *is the ratio of the* DISTANCE *to the* ABSCISSA.
The COSECANT *is the ratio of the* DISTANCE *to the* ORDINATE.

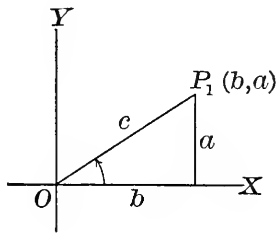
Note. These definitions include those of Art. 10. The definitions of the versed sine and coversed sine, given in Art. 11, are sufficiently general to apply to any angle whatever.

33. We will now apply the definitions of Art. 32 to the angles XOP_1, XOP_2, XOP_3, and XOP_4 in the following figures:

Let P_1, P_2, P_3, and P_4 be any points on the terminal lines OP_1, OP_2, OP_3, and OP_4, and let their co-ordinates be (b, a), $(-b, a)$, $(-b, -a)$, and $(b, -a)$, respectively.

Let $OP_1 = OP_2 = OP_3 = OP_4 = c$.

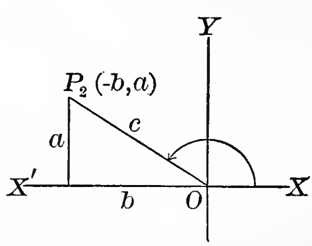
Then by the definitions,



$$\sin XOP_1 = \frac{a}{c}. \qquad \cos XOP_1 = \frac{b}{c}.$$

$$\tan XOP_1 = \frac{a}{b}. \qquad \cot XOP_1 = \frac{b}{a}.$$

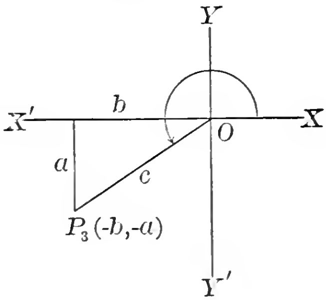
$$\sec XOP_1 = \frac{c}{b}. \qquad \csc XOP_1 = \frac{c}{a}.$$



$$\sin XOP_2 = \frac{a}{c}. \qquad \cos XOP_2 = \frac{-b}{c} = -\frac{b}{c}.$$

$$\tan XOP_2 = \frac{a}{-b} = -\frac{a}{b}. \qquad \cot XOP_2 = \frac{-b}{a} = -\frac{b}{a}.$$

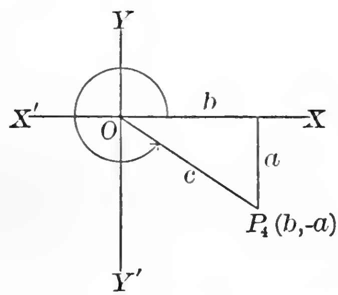
$$\sec XOP_2 = \frac{c}{-b} = -\frac{c}{b}. \qquad \csc XOP_2 = \frac{c}{a}.$$



$$\sin XOP_3 = \frac{-a}{c} = -\frac{a}{c}. \qquad \cos XOP_3 = \frac{-b}{c} = -\frac{b}{c}.$$

$$\tan XOP_3 = \frac{-a}{-b} = \frac{a}{b}. \qquad \cot XOP_3 = \frac{-b}{-a} = \frac{b}{a}.$$

$$\sec XOP_3 = \frac{c}{-b} = -\frac{c}{b}. \qquad \csc XOP_3 = \frac{c}{-a} = -\frac{c}{a}.$$



$$\sin XOP_4 = \frac{-a}{c} = -\frac{a}{c}. \qquad \cos XOP_4 = \frac{b}{c}.$$

$$\tan XOP_4 = \frac{-a}{b} = -\frac{a}{b}. \qquad \cot XOP_4 = \frac{b}{-a} = -\frac{b}{a}.$$

$$\sec XOP_4 = \frac{c}{b}. \qquad \csc XOP_4 = \frac{c}{-a} = -\frac{c}{a}.$$

34. Since the terminal lines of two angles which differ by a multiple of 360° are coincident (Art. 27), it is evident that the trigonometric functions of two such angles are identical.

Thus, the functions of 50°, 410°, 770°, −310°, etc., are identical.

It is customary to express this fact by saying that the trigonometric functions are *periodic*.

35. If the terminal line of an angle lies between OX and OY, the angle is said to be in the *first quadrant;* if between OY and OX', in the *second quadrant;* between OX' and OY', in the *third quadrant;* between OY' and OX, in the *fourth quadrant.*

Thus, any positive angle between 0° and 90°, or between 360° and 450°, or any negative angle between − 270° and − 360°, is in the first quadrant.

Any positive angle between 90° and 180°, or between 450° and 540°, or any negative angle between − 180° and − 270°, is in the second quadrant.

36. We observe, by inspection of the results in Art. 33, the following points in regard to the algebraic signs of the trigonometric functions in the different quadrants:

For any angle in the first quadrant all the functions are positive.

In the second quadrant, the sine and cosecant are positive, and the cosine, tangent, cotangent, and secant are negative.

In the third quadrant, the tangent and cotangent are positive, and the sine, cosine, secant, and cosecant are negative.

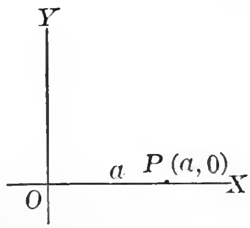
In the fourth quadrant, the cosine and secant are positive, and the sine, tangent, cotangent, and cosecant are negative.

37. It is customary to express the foregoing principles in tabular form, as follows:

Functions.	*First Quad.*	*Second Quad.*	*Third Quad.*	*Fourth Quad.*
Sine and cosecant . . .	+	+	−	−
Cosine and secant . . .	+	−	−	+
Tangent and cotangent .	+	−	+	−

FUNCTIONS OF 0°, 90°, 180°, 270°, AND 360°.

38. *To find the functions of* 0° *and* 360°.



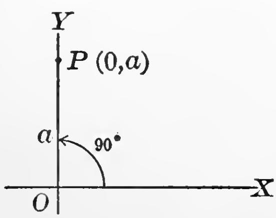
The terminal line of 0° coincides with the initial line OX.
Let P be a point on OX, such that $OP = a$.
Then by Art. 31, the co-ordinates of P are $(a, 0)$.
Whence by definition,

$$\sin 0° = \frac{0}{a} = 0. \qquad \tan 0° = \frac{0}{a} = 0. \qquad \sec 0° = \frac{a}{a} = 1.$$

$$\cos 0° = \frac{a}{a} = 1. \qquad \cot 0° = \frac{a}{0} = \infty. \qquad \csc 0° = \frac{a}{0} = \infty.$$

By Art. 34, the functions of 360° are the same as those of 0°.

39. *To find the functions of* 90°.

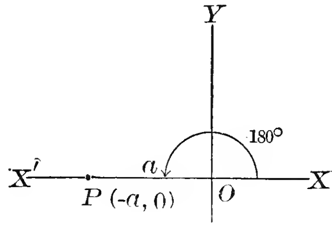


Let P be a point on OY, such that $OP = a$.
Then the co-ordinates of P are $(0, a)$.
Whence by definition,

$$\sin 90° = \frac{a}{a} = 1. \qquad \tan 90° = \frac{a}{0} = \infty. \qquad \sec 90° = \frac{a}{0} = \infty.$$

$$\cos 90° = \frac{0}{a} = 0. \qquad \cot 90° = \frac{0}{a} = 0. \qquad \csc 90° = \frac{a}{a} = 1.$$

40. *To find the functions of* 180°.



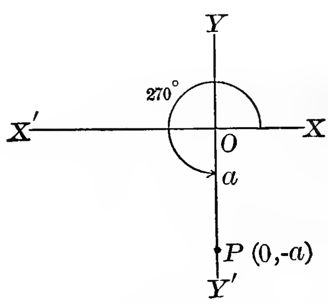
Let P be a point on OX', such that $OP = a$.
Then the co-ordinates of P are $(-a, 0)$.
Whence by definition,

$$\sin 180° = \frac{0}{a} = 0. \qquad \cos 180° = \frac{-a}{a} = -1.$$

$$\tan 180° = \frac{0}{-a} = 0. \qquad \cot 180° = \frac{-a}{0} = \infty.$$

$$\sec 180° = \frac{a}{-a} = -1. \qquad \csc 180° = \frac{a}{0} = \infty.$$

41. *To find the functions of* 270°.



Let P be a point on OY', such that $OP = a$.
Then the co-ordinates of P are $(0, -a)$.
Whence by definition,

$$\sin 270° = \frac{-a}{a} = -1. \qquad \cos 270° = \frac{0}{a} = 0.$$

$$\tan 270° = \frac{-a}{0} = \infty. \qquad \cot 270° = \frac{0}{-a} = 0.$$

$$\sec 270° = \frac{a}{0} = \infty. \qquad \csc 270° = \frac{a}{-a} = -1.$$

Note. No absolute meaning can be attached to such results as $\cot 0^\circ = \infty$, $\tan 90^\circ = \infty$, etc. The equation $\cot 0^\circ = \infty$ merely signifies that as an angle approaches 0° as a limit, its cotangent increases without limit.

A similar interpretation must be given to the equations $\csc 0^\circ = \infty$, $\sec 90^\circ = \infty$, etc.

FUNCTIONS OF $(-A)$ IN TERMS OF THOSE OF A.

42. *To prove the formulæ,*

$$\left.\begin{aligned} \sin(-A) &= -\sin A, & \cos(-A) &= \cos A, \\ \tan(-A) &= -\tan A, & \cot(-A) &= -\cot A, \\ \sec(-A) &= \sec A, & \csc(-A) &= -\csc A, \end{aligned}\right\} \qquad (9)$$

for any value of A.

There will be four cases, according as A is in the first, second, third, or fourth quadrant.

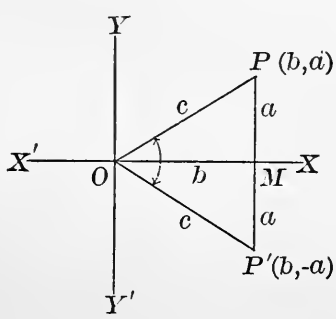
In each figure, let the positive angle XOP (indicated by the *full* arc) represent the angle A, and the negative angle XOP' (indicated by the *dotted* arc) the angle $(-A)$.

Draw PP' perpendicular to XX'; then the right triangles OPM and $OP'M$ have the side OM and the angle POM of one equal to the side OM and the angle $P'OM$ of the other, and are equal.

Whence, $PM = P'M$ and $OP = OP'$.

Let $PM = P'M = a$, $OM = b$, and $OP = OP' = c$.

Case I. *A in the first quadrant, $-A$ in the fourth.*



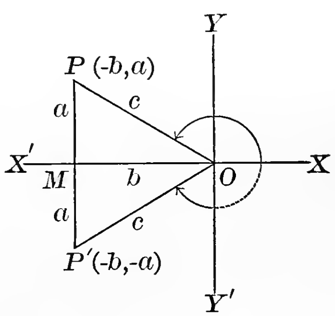
By definition, we have:

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$
$-A$	$-\frac{a}{c}$	$\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$\frac{c}{b}$	$-\frac{c}{a}$

It is evident from the above that the cosine and secant of $-A$ are equal respectively to the cosine and secant of A, while the sine, tangent, cotangent, and cosecant of $-A$ are equal respectively to the negatives of the sine, tangent, cotangent, and cosecant of A.

Whence we obtain the formulæ (9).

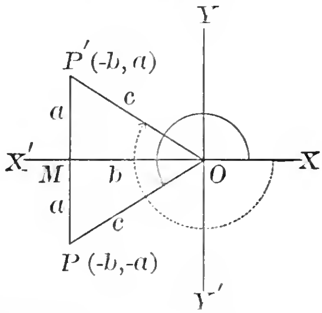
CASE II. *A in the second quadrant, $-A$ in the third.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$-\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$-\frac{c}{b}$	$\frac{c}{a}$
$-A$	$-\frac{a}{c}$	$-\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$-\frac{c}{b}$	$-\frac{c}{a}$

Whence we obtain the formulæ (9) as before.

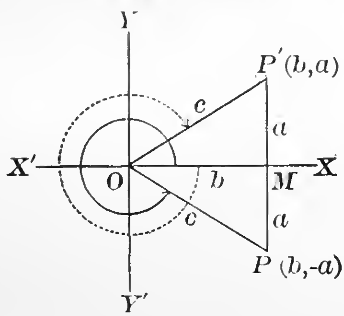
CASE III. *A in the third quadrant, $-A$ in the second.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$-\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$-\frac{c}{b}$	$-\frac{c}{a}$
$-A$	$\frac{a}{c}$	$-\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$-\frac{c}{b}$	$\frac{c}{a}$

Whence we obtain the formulæ (9) as before.

CASE IV. *A in the fourth quadrant, $-A$ in the first.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$\frac{c}{b}$	$-\frac{c}{a}$
$-A$	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$

Whence we obtain the formulæ (9) as before.

43. The results of Art. 42 may be stated as follows:

The sine, cosine, tangent, cotangent, secant, and cosecant of a negative angle are equal respectively to the negative sine, the cosine, the negative tangent, the negative cotangent, the secant, and the negative cosecant, of the absolute value of the angle.

FUNCTIONS OF $(90° + A)$ IN TERMS OF THOSE OF A.

44. *To prove the formulæ,*

$$\left.\begin{aligned} \sin(90° + A) &= \cos A, & \cos(90° + A) &= -\sin A, \\ \tan(90° + A) &= -\cot A, & \cot(90° + A) &= -\tan A, \\ \sec(90° + A) &= -\csc A, & \csc(90° + A) &= \sec A, \end{aligned}\right\} \quad (10)$$

for any value of A.

There will be four cases, according as A is in the first, second, third, or fourth quadrant.

In each figure, let the positive angle XOP (indicated by the *full* arc) represent the angle A, and the positive angle XOP' (indicated by the *dotted* arc) the angle $(90° + A)$.

Lay off $OP = OP'$, and draw PM and $P'M'$ perpendicular to XX'.

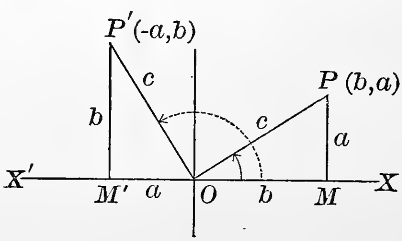
Since OP and OM are perpendicular to OP' and $P'M'$ respectively, the angles POM and $OP'M'$ are equal.

Then the right triangles OPM and $OP'M'$ have the hypotenuse OP and the angle POM of one equal to the hypotenuse OP' and the angle $OP'M'$ of the other, and are equal.

Whence, $PM = OM'$ and $OM = P'M'$.

Let $PM = OM' = a$, $OM = P'M' = b$, and $OP = OP' = c$.

CASE I. *A in the first quadrant*, $90° + A$ *in the second.*



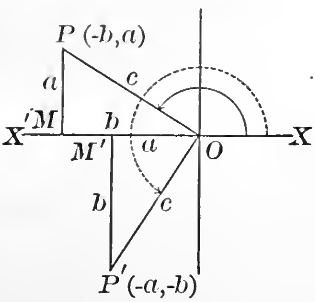
By definition, we have:

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$
$90°+A$	$\frac{b}{c}$	$-\frac{a}{c}$	$-\frac{b}{a}$	$-\frac{a}{b}$	$-\frac{c}{a}$	$\frac{c}{b}$

It is evident from the above that the sine and cosecant of $90° + A$ are equal respectively to the cosine and secant of A, while the cosine, tangent, cotangent, and secant of $90° + A$ are equal respectively to the negatives of the sine, cotangent, tangent, and cosecant of A.

Whence we obtain the formulæ (**10**).

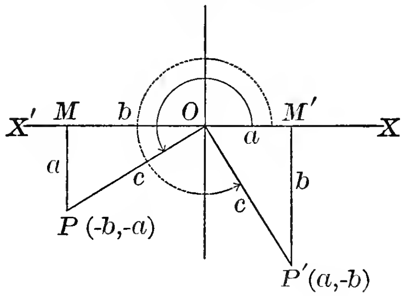
CASE II. *A in the second quadrant, $90° + A$ in the third.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$\frac{a}{c}$	$-\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$-\frac{c}{b}$	$\frac{c}{a}$
$90°+A$	$-\frac{b}{c}$	$-\frac{a}{c}$	$\frac{b}{a}$	$\frac{a}{b}$	$-\frac{c}{a}$	$-\frac{c}{b}$

Whence we obtain the formulæ (**10**) as before.

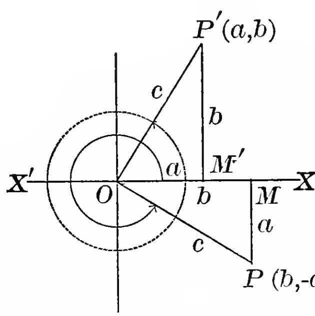
CASE III. *A in the third quadrant*, $90° + A$ *in the fourth.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$-\frac{b}{c}$	$\frac{a}{b}$	$\frac{b}{a}$	$-\frac{c}{b}$	$-\frac{c}{a}$
$90°+A$	$-\frac{b}{c}$	$\frac{a}{c}$	$-\frac{b}{a}$	$-\frac{a}{b}$	$\frac{c}{a}$	$-\frac{c}{b}$

Whence we obtain the formulæ (**10**) as before.

CASE IV. *A in the fourth quadrant*, $90° + A$ *in the first.*



Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
A	$-\frac{a}{c}$	$\frac{b}{c}$	$-\frac{a}{b}$	$-\frac{b}{a}$	$\frac{c}{b}$	$-\frac{c}{a}$
$90°+A$	$\frac{b}{c}$	$\frac{a}{c}$	$\frac{b}{a}$	$\frac{a}{b}$	$\frac{c}{a}$	$\frac{c}{b}$

Whence we obtain the formulæ (**10**) as before.

45. The results of Art. 44 may be stated in the following form:

The sine, cosine, tangent, cotangent, secant, and cosecant of any angle are equal respectively to the cosine, the negative sine, the negative cotangent, the negative tangent, the negative cosecant, and the secant, of an angle 90° less.

46. *To find the values of the functions of* $90° - A$ *in terms of those of* A.

By Arts. 45 and 42, we have:

$$\begin{aligned}
\sin(90° - A) &= \cos(-A) = \cos A.\\
\cos(90° - A) &= -\sin(-A) = \sin A.\\
\tan(90° - A) &= -\cot(-A) = \cot A.\\
\cot(90° - A) &= -\tan(-A) = \tan A.\\
\sec(90° - A) &= -\csc(-A) = \csc A.\\
\csc(90° - A) &= \sec(-A) = \sec A.
\end{aligned}$$

We have thus proved the formulæ of Art. 14 for any value of A.

47. *To find the values of the functions of* $180° - A$ *in terms of those of* A.

By Arts. 45 and 46 we have:

$$\begin{aligned}
\sin(180° - A) &= \cos(90° - A) = \sin A.\\
\cos(180° - A) &= -\sin(90° - A) = -\cos A.\\
\tan(180° - A) &= -\cot(90° - A) = -\tan A.\\
\cot(180° - A) &= -\tan(90° - A) = -\cot A.\\
\sec(180° - A) &= -\csc(90° - A) = -\sec A.\\
\csc(180° - A) &= \sec(90° - A) = \csc A.
\end{aligned}$$

Since the angles A and $180° - A$ are supplements of each other, these formulæ express the values of the functions of the supplement of an angle in terms of those of the angle itself.

48. *To find the values of the functions of* $180° + A$ *in terms of those of* A.

By Arts. 45 and 44, we have:

$$\begin{aligned}
\sin(180°+A) &= \cos(90°+A) = -\sin A.\\
\cos(180°+A) &= -\sin(90°+A) = -\cos A.\\
\tan(180°+A) &= -\cot(90°+A) = \tan A.\\
\cot(180°+A) &= -\tan(90°+A) = \cot A.\\
\sec(180°+A) &= -\csc(90°+A) = -\sec A.\\
\csc(180°+A) &= \sec(90°+A) = -\csc A.
\end{aligned}$$

49. *To find the values of the functions of* $270°-A$ *in terms of those of* A.

By Arts. 45 and 47, we have:

$$\begin{aligned}
\sin(270°-A) &= \cos(180°-A) = -\cos A.\\
\cos(270°-A) &= -\sin(180°-A) = -\sin A.\\
\tan(270°-A) &= -\cot(180°-A) = \cot A.\\
\cot(270°-A) &= -\tan(180°-A) = \tan A.\\
\sec(270°-A) &= -\csc(180°-A) = -\csc A.\\
\csc(270°-A) &= \sec(180°-A) = -\sec A.
\end{aligned}$$

50. *To find the values of the functions of* $270°+A$ *in terms of those of* A.

By Arts. 45 and 48, we have:

$$\begin{aligned}
\sin(270°+A) &= \cos(180°+A) = -\cos A.\\
\cos(270°+A) &= -\sin(180°+A) = \sin A.\\
\tan(270°+A) &= -\cot(180°+A) = -\cot A.\\
\cot(270°+A) &= -\tan(180°+A) = -\tan A.\\
\sec(270°+A) &= -\csc(180°+A) = \csc A.\\
\csc(270°+A) &= \sec(180°+A) = -\sec A.
\end{aligned}$$

51. *To find the values of the functions of $360° - A$ and $360° + A$ in terms of those of A.*

By Art. 34, the functions of $360° - A$ are the same as those of $-A$, and the functions of $360° + A$ are the same as those of A.

Whence by Art. 42,

$$\sin(360° - A) = -\sin A. \qquad \cos(360° - A) = \cos A.$$
$$\tan(360° - A) = -\tan A. \qquad \cot(360° - A) = -\cot A.$$
$$\sec(360° - A) = \sec A. \qquad \csc(360° - A) = -\csc A.$$

And,

$$\sin(360° + A) = \sin A. \qquad \cos(360° + A) = \cos A.$$
$$\tan(360° + A) = \tan A. \qquad \cot(360° + A) = \cot A.$$
$$\sec(360° + A) = \sec A. \qquad \csc(360° + A) = \csc A.$$

52. The operation illustrated in Art. 51 may be extended indefinitely; we should find for the angles $450° - A$, $810° - A$, $-270° - A$, etc., the same results as for $90° - A$; and so on.

53. The results of Arts. 42 to 52 may be expressed in the following rule, which is derived by inspection from the formulæ of Arts. 42 to 51:

Any function of 0°, *or an* EVEN *multiple of* 90°, *plus or minus A, is the* SAME *function of A; and any function of an* ODD *multiple of* 90°, *plus or minus A, is the* COMPLEMENTARY *function of A.*

To obtain the algebraic sign, regard A as an acute angle, and apply the table of Art. 37.

1. Required the value of $\sec(990° + A)$.

Since 990° is an odd multiple of 90°, the absolute value of the result is $\csc A$.

And if A is an acute angle, $990° + A$ is in the fourth quadrant, in which the sign of the secant is positive.

Hence, $$\sec(990° + A) = \csc A.$$

2. Required the value of $\tan(-180° - A)$.

Since 180° is an even multiple of 90°, the absolute value of the result is $\tan A$.

And if A is an acute angle, $-180° - A$ is in the second quadrant, in which the sign of the tangent is negative.

Hence, $$\tan(-180° - A) = -\tan A.$$

EXAMPLES.

Find the values of the functions of the following angles in terms of the functions of A:

3. $450° - A$.	**7.** $630° - A$.	**11.** $-180° + A$.
4. $450° + A$.	**8.** $900° - A$.	**12.** $-180° - A$.
5. $540° - A$.	**9.** $-90° + A$.	**13.** $-270° + A$.
6. $540° + A$.	**10.** $-90° - A$.	**14.** $-720° + A$.

54. The functions of any positive or negative angle whatever may be expressed in terms of the functions of an angle between 0° and 90°.

1. Express $\sin 317°$ as a function of an angle between 0° and 90°.

We have by the rule of Art. 53,

$$\sin 317° = \sin(270° + 47°) = -\cos 47°,$$

or $$\sin 317° = \sin(360° - 43°) = -\sin 43°.$$

EXAMPLES.

Express the following in terms of the functions of angles between 0° and 90°:

2. $\cos 152°$.	**4.** $\sec(-77°)$.	**6.** $\cot(-129°)$.
3. $\tan 522°$.	**5.** $\csc 230°$.	**7.** $\sin 865°$.

Express the following in terms of the functions of angles between 0° and 45°:

8. $\cot 83°$.	**10.** $\sec 165°$.	**12.** $\tan 520°$.
9. $\sin(-50°)$.	**11.** $\cos(-303°)$.	**13.** $\csc 768°$.

55. The following table gives, for convenient reference, the functions of the angles 0°, 30°, 45°, 60°, 90°, 120°, etc.

The values of the functions of 0°, 30°, 45°, 60°, 90°, 180°, 270°, and 360° have already been proved in Arts. 16. 17, 38, 39, 40, and 41; the others may be derived by aid of Art. 53, and are left as exercises for the student.

As an illustration, we will give the proof for cot 150°.

By the rule of Art. 53:

$\cot 150° = \cot(180° - 30°) = - \cot 30° = -\sqrt{3}$ (Art. 17).

Angle.	*Sin.*	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	1	0	∞	0	∞	1
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
180°	0	-1	0	∞	-1	∞
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
270°	-1	0	∞	0	∞	-1
300°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
360°	0	1	0	∞	1	∞

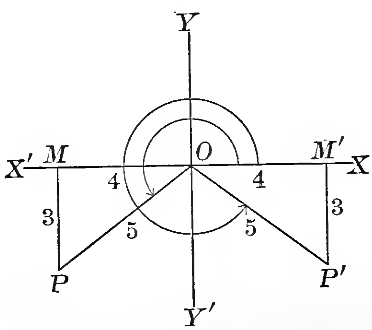
56. *Given the value of one of the functions of an angle, to find the values of the remaining functions.* (Compare Art. 15.)

1. Given $\sin A = -\frac{3}{5}$; required the values of the remaining functions of A.

The example may be solved by a method similar to that employed in Art. 15.

Since the sine is the ratio of the ordinate to the distance, we may regard the point of reference as having its ordinate equal to -3, and its distance equal to 5.

There are *two* such points, P and P', and consequently *two angles*, XOP and XOP', in the third and fourth quadrants respectively, either of which satisfies the given condition.



Then $OM = OM' = \sqrt{\overline{OP}^2 - \overline{PM}^2} = \sqrt{25-9} = 4$, and the co-ordinates of P are $(-4, -3)$, and of P', $(4, -3)$.

Whence, $\cos XOP = -\frac{4}{5}$, $\cos XOP' = \frac{4}{5}$,

$\tan XOP = \frac{3}{4}$, $\tan XOP' = -\frac{3}{4}$,

$\cot XOP = \frac{4}{3}$, $\cot XOP' = -\frac{4}{3}$,

$\sec XOP = -\frac{5}{4}$, $\sec XOP' = \frac{5}{4}$,

$\csc XOP = -\frac{5}{3}$, $\csc XOP' = -\frac{5}{3}$.

Thus the two solutions to the example are:

$$\cos A = \mp\frac{4}{5}, \quad \tan A = \pm\frac{3}{4}, \quad \cot A = \pm\frac{4}{3},$$

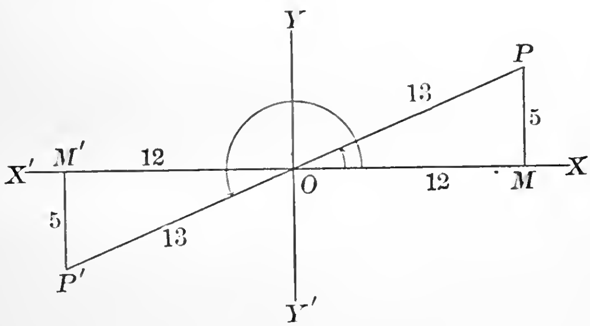
$$\sec A = \mp\frac{5}{4}, \quad \csc A = -\frac{5}{3},$$

where the upper signs refer to the angle XOP, and the lower signs to XOP'.

2. Given $\cot A = \frac{12}{5}$; required the values of the remaining functions of A.

The equation may be written $\cot A = \frac{-12}{-5}$.

We may then regard the point of reference as having its abscissa equal to 12 and its ordinate equal to 5, or as having its abscissa equal to -12 and its ordinate equal to -5; and there are consequently *two angles*, XOP and XOP', in the first and third quadrants respectively, either of which satisfies the given condition.



Then $OP = OP' = \sqrt{\overline{OM}^2 + \overline{PM}^2} = \sqrt{144 + 25} = 13.$

$$\text{Whence, } \sin XOP = \frac{5}{13}, \qquad \sin XOP' = -\frac{5}{13},$$

$$\cos XOP = \frac{12}{13}, \qquad \cos XOP' = -\frac{12}{13},$$

$$\tan XOP = \frac{5}{12}, \qquad \tan XOP' = \frac{5}{12},$$

$$\sec XOP = \frac{13}{12}, \qquad \sec XOP' = -\frac{13}{12},$$

$$\csc XOP = \frac{13}{5}, \qquad \csc XOP' = -\frac{13}{5}.$$

Thus the two solutions are,

$$\sin A = \pm\frac{5}{13}, \quad \cos A = \pm\frac{12}{13}, \quad \tan A = \frac{5}{12},$$
$$\sec A = \pm\frac{13}{12}, \quad \csc A = \pm\frac{13}{5}.$$

EXAMPLES.

In each case find the values of the remaining functions:

3. $\sin A = \frac{1}{4}$.

4. $\cot A = 2$.

5. $\csc A = -\frac{3}{2}$.

6. $\tan A = -\frac{8}{15}$.

7. $\sec A = \frac{4}{3}$.

8. $\cos A = -\frac{1}{2}$.

9. $\csc A = \sqrt{2}$.

10. $\tan A = 2\sqrt{2}$.

11. $\cos A = -\frac{a}{b}$.

12. $\sin A = x$.

13. $\cot A = \frac{1}{x}$.

14. $\sec A = \frac{\sqrt{a^2+b^2}}{a}$.

PROOFS OF THE FUNDAMENTAL FORMULÆ FOR ANY ANGLE.

57. We have from the general definitions of Art. 32,

$$\sin A \times \csc A = \frac{\text{ordinate}}{\text{distance}} \times \frac{\text{distance}}{\text{ordinate}} = 1.$$

$$\cos A \times \sec A = \frac{\text{abscissa}}{\text{distance}} \times \frac{\text{distance}}{\text{abscissa}} = 1.$$

$$\tan A \times \cot A = \frac{\text{ordinate}}{\text{abscissa}} \times \frac{\text{abscissa}}{\text{ordinate}} = 1.$$

Whence we obtain, as in Art. 18,

$$\sin A = \frac{1}{\csc A}, \quad \tan A = \frac{1}{\cot A}, \quad \sec A = \frac{1}{\cos A},$$

$$\cos A = \frac{1}{\sec A}, \quad \cot A = \frac{1}{\tan A}, \quad \csc A = \frac{1}{\sin A}.$$

58. *To prove the formulæ,*

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A, \quad \csc^2 A = 1 + \cot^2 A,$$

for any value of A.

Since the distance of the point of reference is the hypotenuse of a right triangle whose sides are equal to the absolute values of the abscissa and ordinate, we have by Geometry,

$$(\text{ordinate})^2 + (\text{abscissa})^2 = (\text{distance})^2.$$

This may be written in the forms,

$$\left(\frac{\text{ordinate}}{\text{distance}}\right)^2 + \left(\frac{\text{abscissa}}{\text{distance}}\right)^2 = 1,$$

$$\left(\frac{\text{distance}}{\text{abscissa}}\right)^2 = 1 + \left(\frac{\text{ordinate}}{\text{abscissa}}\right)^2,$$

and

$$\left(\frac{\text{distance}}{\text{ordinate}}\right)^2 = 1 + \left(\frac{\text{abscissa}}{\text{ordinate}}\right)^2.$$

Hence for any value of A, we have

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A, \quad \csc^2 A = 1 + \cot^2 A.$$

59. *To prove the formulæ*

$$\tan A = \frac{\sin A}{\cos A} \quad \textit{and} \quad \cot A = \frac{\cos A}{\sin A},$$

for any value of A.

By definition,

$$\tan A = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\dfrac{\text{ordinate}}{\text{distance}}}{\dfrac{\text{abscissa}}{\text{distance}}} = \frac{\sin A}{\cos A},$$

and

$$\cot A = \frac{\text{abscissa}}{\text{ordinate}} = \frac{\dfrac{\text{abscissa}}{\text{distance}}}{\dfrac{\text{ordinate}}{\text{distance}}} = \frac{\cos A}{\sin A}.$$

IV. MISCELLANEOUS THEOREMS.

TO EXPRESS EACH OF THE SIX PRINCIPAL FUNCTIONS IN TERMS OF THE OTHER FIVE.

60. The following table expresses the value of each of the six principal functions in terms of the other five:

sin	. . .	$\sqrt{1-\cos^2}$	$\frac{\tan}{\sqrt{1+\tan^2}}$	$\frac{1}{\sqrt{1+\cot^2}}$	$\frac{\sqrt{\sec^2-1}}{\sec}$	$\frac{1}{\csc}$
cos	$\sqrt{1-\sin^2}$	. . .	$\frac{1}{\sqrt{1+\tan^2}}$	$\frac{\cot}{\sqrt{1+\cot^2}}$	$\frac{1}{\sec}$	$\frac{\sqrt{\csc^2-1}}{\csc}$
tan	$\frac{\sin}{\sqrt{1-\sin^2}}$	$\frac{\sqrt{1-\cos^2}}{\cos}$	. . .	$\frac{1}{\cot}$	$\sqrt{\sec^2-1}$	$\frac{1}{\sqrt{\csc^2-1}}$
cot	$\frac{\sqrt{1-\sin^2}}{\sin}$	$\frac{\cos}{\sqrt{1-\cos^2}}$	$\frac{1}{\tan}$	. . .	$\frac{1}{\sqrt{\sec^2-1}}$	$\sqrt{\csc^2-1}$
sec	$\frac{1}{\sqrt{1-\sin^2}}$	$\frac{1}{\cos}$	$\sqrt{1+\tan^2}$	$\frac{\sqrt{1+\cot^2}}{\cot}$	. . .	$\frac{\csc}{\sqrt{\csc^2-1}}$
csc	$\frac{1}{\sin}$	$\frac{1}{\sqrt{1-\cos^2}}$	$\frac{\sqrt{1+\tan^2}}{\tan}$	$\sqrt{1+\cot^2}$	$\frac{\sec}{\sqrt{\sec^2-1}}$	. . .

The reciprocal forms were proved in Art. 57; the others may be derived by aid of Arts. 57, 58, and 59, and are left as exercises for the student.

As an illustration, we will give a proof of the formula

$$\cos A = \frac{\sqrt{\csc^2 A - 1}}{\csc A}.$$

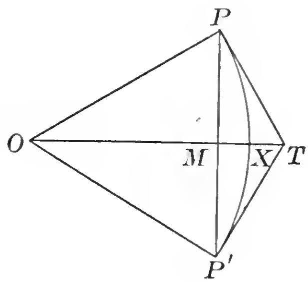
By Art. 58, $$\cos A = \sqrt{1-\sin^2 A}$$

$$= \sqrt{1 - \frac{1}{\csc^2 A}} = \frac{\sqrt{\csc^2 A - 1}}{\csc A}.$$

LIMITING VALUES OF $\frac{\sin x}{x}$ AND $\frac{\tan x}{x}$.

61. *To find the limiting values of the fractions* $\frac{\sin x}{x}$ *and* $\frac{\tan x}{x}$, *as* x *approaches the limit* 0.

Note. We suppose x to be expressed in *circular measure* (Art. 4).



Let $OPXP'$ be a circular sector; draw PT and $P'T$ tangent to the arc at P and P', and join OT and PP'.

By Geometry, $PT = P'T$.

Then OT is perpendicular to PP' at its middle point M, and bisects the arc PP' at X.

Let $\angle\ XOP = \angle\ XOP' = x$.

By Geometry, $\quad$ arc $PP' >$ chord PP', and $< PTP'$.

Whence, $\quad$ arc $PX > PM$, and $< PT$.

Therefore, $\quad \frac{\text{arc } PX}{OP} > \frac{PM}{OP}$, and $< \frac{PT}{OP}$,

or, by Art. 5, $\quad$ circ. meas. $x > \sin x$, and $< \tan x$. $\quad$ (A)

Representing the circular measure of x by x simply, and dividing through by $\sin x$, we have

$$\frac{x}{\sin x} > 1, \text{ and } < \frac{\tan x}{\sin x} \text{ or } \frac{1}{\cos x}.$$

That is, $$\frac{\sin x}{x} < 1, \text{ and } > \cos x.$$

But as x approaches the limit 0, $\cos x$ approaches the limit 1 (Art. 38).

Hence, $\dfrac{\sin x}{x}$ approaches the limit 1 as x approaches 0.

Again,
$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \frac{\sin x}{x} \times \frac{1}{\cos x}.$$

But each factor approaches the limit 1 as x approaches 0.

Hence, $\dfrac{\tan x}{x}$ approaches the limit 1 as x approaches 0.

LINE VALUES OF THE TRIGONOMETRIC FUNCTIONS.

62. Let AOB be any angle.

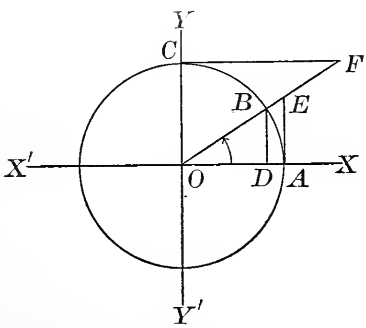


FIG. 1.

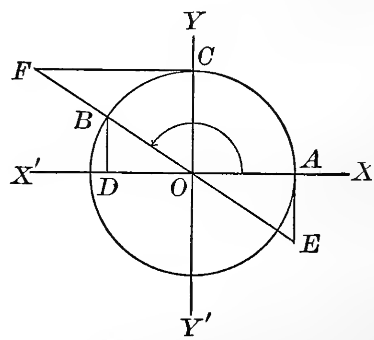


FIG. 2.

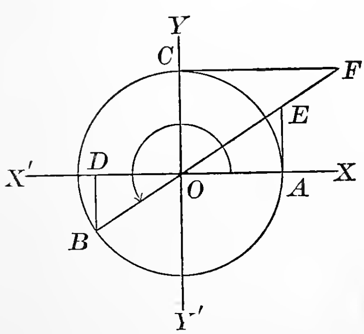


FIG. 3.

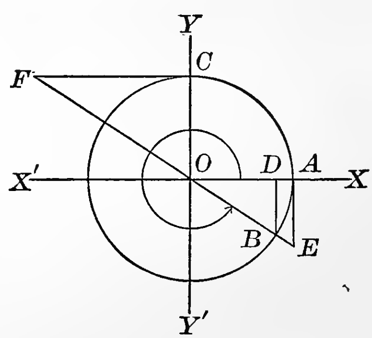


FIG. 4.

With O as a centre, and a radius equal to 1, describe the circle AB; draw BD and AE perpendicular to XX', and CF perpendicular to YY'.

Then by Art. 32, the functions of AOB are:

	Sin.	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
Fig. 1	$\frac{BD}{OB}$	$\frac{OD}{OB}$	$\frac{BD}{OD}$	$\frac{OD}{BD}$	$\frac{OB}{OD}$	$\frac{OB}{BD}$
Fig. 2	$\frac{BD}{OB}$	$-\frac{OD}{OB}$	$-\frac{BD}{OD}$	$-\frac{OD}{BD}$	$-\frac{OB}{OD}$	$\frac{OB}{BD}$
Fig. 3	$-\frac{BD}{OB}$	$-\frac{OD}{OB}$	$\frac{BD}{OD}$	$\frac{OD}{BD}$	$-\frac{OB}{OD}$	$-\frac{OB}{BD}$
Fig. 4	$-\frac{BD}{OB}$	$\frac{OD}{OB}$	$-\frac{BD}{OD}$	$-\frac{OD}{BD}$	$\frac{OB}{OD}$	$-\frac{OB}{BD}$

But since the right triangles OBD, OEA, and OCF are similar, and $OA = OC = 1$, we have,

$$\frac{BD}{OD} = \frac{AE}{OA} = AE, \qquad \frac{OB}{OD} = \frac{OE}{OA} = OE,$$

$$\frac{OD}{BD} = \frac{CF}{OC} = CF, \qquad \frac{OB}{BD} = \frac{OF}{OC} = OF.$$

Whence, since $OB = 1$, the functions of AOB are:

	Sin.	*Cos.*	*Tan.*	*Cot.*	*Sec.*	*Csc.*
Fig. 1	BD	OD	AE	CF	OE	OF
Fig. 2	BD	$-OD$	$-AE$	$-CF$	$-OE$	OF
Fig. 3	$-BD$	$-OD$	AE	CF	$-OE$	$-OF$
Fig. 4	$-BD$	OD	$-AE$	$-CF$	OE	$-OF$

That is, *if the radius of the circle is unity*,

The *sine* is the perpendicular drawn to XX' from the intersection of the circle with the terminal line.

The *cosine* is the line drawn from the centre to the foot of the sine.

The *tangent* is that portion of the geometrical tangent to the circle at its intersection with OX included between OX and the terminal line, produced if necessary.

The *cotangent* is that portion of the geometrical tangent to the circle at its intersection with OY included between OY and the terminal line, produced if necessary.

The *secant* is that portion of the terminal line, or terminal line produced, included between the centre and the tangent.

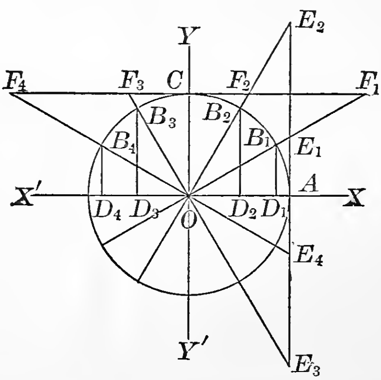
The *cosecant* is that portion of the terminal line, or terminal line produced, included between the centre and the cotangent.

And with regard to algebraic signs,

Sines and tangents measured *above* XX' are *positive*, and *below*, *negative*; cosines and cotangents measured to the *right* of YY' are *positive*, and to the *left*, *negative*; secants and cosecants measured on the terminal line itself are *positive*, and on the terminal line *produced*, *negative*.

63. The above are called the *line values* of the trigonometric functions. They simply *represent* the values of the functions when the radius is unity; that is, the *numerical value* of the sine of an angle is the same as the *number* which expresses the length of the perpendicular drawn to XX' from the intersection of the circle and terminal line.

64. *To trace the changes in the six principal trigonometric functions of an angle as the angle increases from* 0° *to* 360°.



Let the terminal line start from the position OA, and revolve about O as a pivot in the direction of OY, occupying successively the positions OB_1, OB_2, OC, OB_3, OB_4, etc.

Then since the sine of the angle commences with the value 0, and assumes in succession the values B_1D_1, B_2D_2, OC, B_3D_3, B_4D_4, etc. (Art. 60), it is evident that as the angle increases from 0° to 90°, the sine increases from 0 to 1; from 90° to 180°, it decreases from 1 to 0; from 180° to 270°, it decreases (algebraically) from 0 to -1; and from 270° to 360°, it increases from -1 to 0.

Since the cosine commences with the value OA, and assumes in succession the values OD_1, OD_2, 0, $-OD_3$, $-OD_4$, etc., from 0° to 90°, it decreases from 1 to 0; from 90° to 180°, it decreases from 0 to -1; from 180° to 270°, it increases from -1 to 0; and from 270° to 360°, it increases from 0 to 1.

Since the tangent commences with the value 0, and assumes in succession the values AE_1, AE_2, ∞, $-AE_3$, $-AE_4$, etc., from 0° to 90°, it increases from 0 to ∞; from 90° to 180°, it increases from $-\infty$ to 0; from 180° to 270°, it increases from 0 to ∞; and from 270° to 360°, it increases from $-\infty$ to 0.

Since the cotangent commences at ∞, and assumes in succession the values CF_1, CF_2, 0, $-CF_3$, $-CF_4$, etc., from 0° to 90°, it decreases from ∞ to 0; from 90° to 180°, it decreases from 0 to $-\infty$; from 180° to 270°, it decreases from ∞ to 0; and from 270° to 360°, it decreases from 0 to $-\infty$.

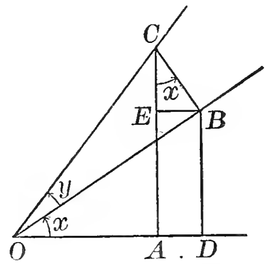
Since the secant commences at OA, and assumes in succession the values OE_1, OE_2, ∞, $-OE_3$, $-OE_4$, etc., from 0° to 90°, it increases from 1 to ∞; from 90° to 180°, it increases from $-\infty$ to -1; from 180° to 270°, it decreases from -1 to $-\infty$; and from 270° to 360°, it decreases from ∞ to 1.

Since the cosecant commences at ∞, and assumes in succession the values OF_1, OF_2, OC, OF_3, OF_4, etc., from 0° to 90°, it decreases from ∞ to 1; from 90° to 180°, it increases from 1 to ∞; from 180° to 270°, it increases from $-\infty$ to -1; and from 270° to 360°, it decreases from -1 to $-\infty$.

Note. Wherever the symbol ∞ occurs in the foregoing discussion, it must be interpreted in accordance with the Note to Art. 41.

V. GENERAL FORMULÆ.

65. *To find the values of* $\sin(x+y)$ *and* $\cos(x+y)$ *in terms of the sines and cosines of* x *and* y.



Let AOB and BOC denote the angles x and y, respectively; then, $AOC = x + y$.

From any point C in OC draw CA and CB perpendicular to OA and OB; and draw BD and BE perpendicular to OA and AC.

Since EC and BC are perpendicular to OA and OB, the angles BCE and AOB are equal; that is, $BCE = x$.

We then have

$$\sin(x+y) = \frac{AC}{OC} = \frac{BD + CE}{OC} = \frac{BD}{OC} + \frac{CE}{OC}.$$

But $$\frac{BD}{OC} = \frac{BD}{OB} \times \frac{OB}{OC} = \sin x \cos y,$$

and $$\frac{CE}{OC} = \frac{CE}{BC} \times \frac{BC}{OC} = \cos x \sin y.$$

Whence, $$\sin(x+y) = \sin x \cos y + \cos x \sin y. \tag{11}$$

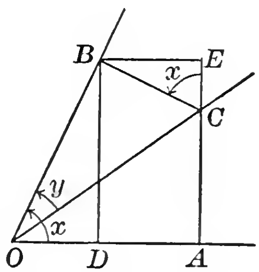
Again, $$\cos(x+y) = \frac{OA}{OC} = \frac{OD - BE}{OC} = \frac{OD}{OC} - \frac{BE}{OC}.$$

But $$\frac{OD}{OC} = \frac{OD}{OB} \times \frac{OB}{OC} = \cos x \cos y,$$

and $$\frac{BE}{OC} = \frac{BE}{BC} \times \frac{BC}{OC} = \sin x \sin y.$$

Whence, $$\cos(x+y) = \cos x \cos y - \sin x \sin y. \tag{12}$$

66. *To find the values of* $\sin(x-y)$ *and* $\cos(x-y)$ *in terms of the sines and cosines of* x *and* y.



Let AOB and BOC denote the angles x and y, respectively; then, $AOC = x - y$.

From any point C in OC draw CA and CB perpendicular to OA and OB; also, draw BD perpendicular to OA, and BE perpendicular to AC produced.

Since EC and BC are perpendicular to OA and OB, the angles BCE and AOB are equal; that is, $BCE = x$.

We then have

$$\sin(x-y) = \frac{AC}{OC} = \frac{BD - CE}{OC} = \frac{BD}{OC} - \frac{CE}{OC}.$$

But $$\frac{BD}{OC} = \frac{BD}{OB} \times \frac{OB}{OC} = \sin x \cos y,$$

and $$\frac{CE}{OC} = \frac{CE}{BC} \times \frac{BC}{OC} = \cos x \sin y.$$

Whence, $$\sin(x-y) = \sin x \cos y - \cos x \sin y. \qquad (13)$$

Again, $$\cos(x-y) = \frac{OA}{OC} = \frac{OD + BE}{OC} = \frac{OD}{OC} + \frac{BE}{OC}.$$

But $$\frac{OD}{OC} = \frac{OD}{OB} \times \frac{OB}{OC} = \cos x \cos y,$$

and $$\frac{BE}{OC} = \frac{BE}{BC} \times \frac{BC}{OC} = \sin x \sin y.$$

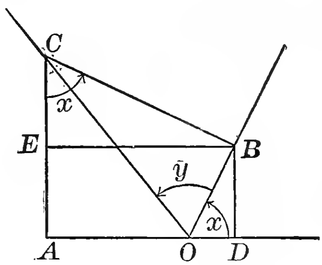
Whence, $$\cos(x-y) = \cos x \cos y + \sin x \sin y. \qquad (14)$$

67. The fundamental formulæ of Arts. 65 and 66 are of great importance, and it is necessary to prove that they hold for all values of x and y.

It is obvious that the proof of Art. 65 is not general, for we have assumed in the construction of the figure that x and y are acute angles, and that $x+y$ is $< 90°$. Also, in Art. 66, we have taken x and y as acute angles, and $x > y$.

In order to prove the formulæ universally, we will first show that (**11**) and (**12**) hold for all values of x and y, and we can then give a general proof of (**13**) and (**14**).

68. We will first prove (**11**) and (**12**) when x and y are acute, and $x+y > 90°$.



Let DOB and BOC denote the angles x and y, respectively; then, $DOC = x + y$.

From any point C in OC draw CB perpendicular to OB, and CA perpendicular to OD produced; and draw BD and BE perpendicular to OD and AC.

Since EC and BC are perpendicular to OD and OB, the angles BCE and DOB are equal; that is, $BCE = x$.

We then have, by Art. 32,

$$\sin DOC = \frac{AC}{OC} = \frac{BD + CE}{OC} = \frac{BD}{OC} + \frac{CE}{OC}.$$

But $$\frac{BD}{OC} = \frac{BD}{OB} \times \frac{OB}{OC} = \sin x \cos y,$$

and $$\frac{CE}{OC} = \frac{CE}{BC} \times \frac{BC}{OC} = \cos x \sin y.$$

Whence, $\sin(x+y) = \sin x \cos y + \cos x \sin y.$

Again, $\cos DOC = \dfrac{-OA}{OC} = \dfrac{OD - BE}{OC} = \dfrac{OD}{OC} - \dfrac{BE}{OC}.$

But $$\frac{OD}{OC} = \frac{OD}{OB} \times \frac{OB}{OC} = \cos x \cos y,$$

and $$\frac{BE}{OC} = \frac{BE}{BC} \times \frac{BC}{OC} = \sin x \sin y.$$

Whence, $\cos(x+y) = \cos x \cos y - \sin x \sin y.$

69. We have thus proved (**11**) and (**12**) when x and y are *any two acute angles;* or, what is the same thing, when they are any two angles in the *first quadrant.*

Now let a and b be any assigned values of x and y for which (**11**) and (**12**) are true; then by Art. 44,

$$\sin[90° + (a+b)] = \cos(a+b)$$
$$= \cos a \cos b - \sin a \sin b, \quad \text{by } (\mathbf{12}); \qquad (\mathbf{A})$$

and,

$$\cos[90° + (a+b)] = -\sin(a+b)$$
$$= -\sin a \cos b - \cos a \sin b, \text{ by} (\mathbf{11}). \qquad (\mathbf{B})$$

But by Art. 44,

$$\cos a = \sin(90° + a), \qquad \cos b = \sin(90° + b),$$
$$-\sin a = \cos(90° + a), \qquad -\sin b = \cos(90° + b).$$

Hence (**A**) may be written in the forms,

$$\sin[(90° + a) + b] = \sin(90° + a)\cos b + \cos(90° + a)\sin b,$$
$$\sin[a + (90° + b)] = \sin a \cos(90° + b) + \cos a \sin(90° + b),$$

both of which are in accordance with (**11**).

And (**B**) may be written in the forms,

$$\cos[(90° + a) + b] = \cos(90° + a)\cos b - \sin(90° + a)\sin b,$$
$$\cos[a + (90° + b)] = \cos a \cos(90° + b) - \sin a \sin(90° + b),$$

both of which are in accordance with (**12**).

It follows from the above that if (**11**) and (**12**) hold for any assigned values of x and y, such as a and b, they also hold when either a or b is increased by 90°.

But they have been proved to hold when both x and y are in the first quadrant; hence they also hold when x is in the second quadrant and y in the first. And since they hold when x is in the second quadrant and y in the first, they also hold when x is in the third quadrant and y in the first; and so on.

Thus (**11**) and (**12**) are proved to hold for any values of x and y whatever, positive or negative.

70. We may now give a general proof of (**13**) and (**14**). By (**11**) and (**12**), we have

$$\begin{aligned}\sin(x-y) &= \sin[x+(-y)]\\ &= \sin x\cos(-y)+\cos x\sin(-y)\\ &= \sin x\cos y-\cos x\sin y \quad \text{(Art. 42)}.\end{aligned}$$

And, $$\begin{aligned}\cos(x-y) &= \cos[x+(-y)]\\ &= \cos x\cos(-y)-\sin x\sin(-y)\\ &= \cos x\cos y+\sin x\sin y \quad \text{(Art. 42)}.\end{aligned}$$

Hence (**13**) and (**14**) hold for all values of x and y, for the above proof depends on formulæ which have been shown to hold universally.

71. By Art. 20, we have

$$\begin{aligned}\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)}\\ &= \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}, \text{ by } (\mathbf{11}) \text{ and } (\mathbf{12}).\end{aligned}$$

Dividing each term of the fraction by $\cos x\cos y$,

$$\tan(x+y)=\frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}}$$

$$=\frac{\tan x+\tan y}{1-\tan x\tan y}. \qquad (\mathbf{15})$$

In like manner, we derive

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}. \qquad \textbf{(16)}$$

Again, by Art. 20, we have

$$\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}, \text{ by } \textbf{(11)} \text{ and } \textbf{(12)}.$$

Dividing each term of the fraction by $\sin x\sin y$,

$$\cot(x+y)=\frac{\dfrac{\cos x\cos y}{\sin x\sin y}-\dfrac{\sin x\sin y}{\sin x\sin y}}{\dfrac{\sin x\cos y}{\sin x\sin y}+\dfrac{\cos x\sin y}{\sin x\sin y}}$$

$$=\frac{\cot x\cot y-1}{\cot y+\cot x}. \qquad \textbf{(17)}$$

In like manner, we derive

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}. \qquad \textbf{(18)}$$

72. From **(11)**, **(12)**, **(13)**, and **(14)**, we obtain

$$\sin(a+b)=\sin a\cos b+\cos a\sin b,$$
$$\sin(a-b)=\sin a\cos b-\cos a\sin b,$$
$$\cos(a+b)=\cos a\cos b-\sin a\sin b,$$
$$\cos(a-b)=\cos a\cos b+\sin a\sin b.$$

Whence, by addition and subtraction,

$$\sin(a+b)+\sin(a-b)=2\sin a\cos b,$$
$$\sin(a+b)-\sin(a-b)=2\cos a\sin b,$$
$$\cos(a+b)+\cos(a-b)=2\cos a\cos b,$$
$$\cos(a+b)-\cos(a-b)=-2\sin a\sin b.$$

Let $a+b=x$, and $a-b=y$.

Then, $a=\frac{1}{2}(x+y)$, and $b=\frac{1}{2}(x-y)$.

Substituting these values, we have

$$\sin x+\sin y=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y), \tag{19}$$

$$\sin x-\sin y=2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y), \tag{20}$$

$$\cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y), \tag{21}$$

$$\cos x-\cos y=-2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y). \tag{22}$$

73. Dividing (**19**) by (**20**), we obtain

$$\frac{\sin x+\sin y}{\sin x-\sin y}=\frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)}$$

$$=\tan\tfrac{1}{2}(x+y)\cot\tfrac{1}{2}(x-y)$$

$$=\frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}\quad\text{(Art. 18)}. \tag{23}$$

FUNCTIONS OF $2x$.

74. Putting $y=x$ in (**11**), we have

$$\sin 2x=\sin x\cos x+\cos x\sin x$$

$$=2\sin x\cos x. \tag{24}$$

Putting $y=x$ in (**12**),

$$\cos 2x=\cos^2 x-\sin^2 x. \tag{25}$$

Since $\cos^2 x=1-\sin^2 x$, and $\sin^2 x=1-\cos^2 x$ (Art. 19), we also have

$$\cos 2x=1-\sin^2 x-\sin^2 x\quad=1-2\sin^2 x, \tag{26}$$

and

$$\cos 2x=\cos^2 x-(1-\cos^2 x)=2\cos^2 x-1. \tag{27}$$

In like manner, from (**15**) and (**17**), we have

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}, \tag{28}$$

$$\cot 2x=\frac{\cot^2 x-1}{2\cot x}. \tag{29}$$

FUNCTIONS OF $\frac{1}{2}x$.

75. From (**26**) and (**27**), we have

$$2\sin^2 x = 1 - \cos 2x, \text{ and } 2\cos^2 x = 1 + \cos 2x.$$

Writing x in place of $2x$, and therefore $\frac{1}{2}x$ in place of x,

$$2\sin^2 \tfrac{1}{2}x = 1 - \cos x, \text{ and } 2\cos^2 \tfrac{1}{2}x = 1 + \cos x. \qquad \text{(A)}$$

Dividing by 2, and extracting the square root,

$$\sin \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{2}}, \qquad (30)$$

$$\cos \tfrac{1}{2}x = \sqrt{\frac{1 + \cos x}{2}}. \qquad (31)$$

Dividing (**30**) by (**31**), we obtain

$$\tan \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}. \qquad (32)$$

Multiplying the terms of the fraction under the radical sign first by $1 + \cos x$, and then by $1 - \cos x$, we have

$$\tan \tfrac{1}{2}x = \sqrt{\frac{1 - \cos^2 x}{(1 + \cos x)^2}}$$

$$= \sqrt{\frac{\sin^2 x}{(1 + \cos x)^2}} = \frac{\sin x}{1 + \cos x}, \qquad (33)$$

and

$$\tan \tfrac{1}{2}x = \sqrt{\frac{(1 - \cos x)^2}{1 - \cos^2 x}}$$

$$= \sqrt{\frac{(1 - \cos x)^2}{\sin^2 x}} = \frac{1 - \cos x}{\sin x}. \qquad (34)$$

And since the cotangent is the reciprocal of the tangent,

$$\cot \tfrac{1}{2}x = \frac{1 + \cos x}{\sin x} = \frac{\sin x}{1 - \cos x}. \qquad (35)$$

Note. The radical in each of the formulæ (**30**), (**31**), and (**32**) is to be taken as positive or negative according to the quadrant in which the angle $\frac{1}{2}x$ is situated (Art. 37).

INVERSE TRIGONOMETRIC FUNCTIONS.

76. The expression $\sin^{-1} y$, called the *inverse sine* of y, or the *anti-sine* of y, is used to denote the angle whose sine is equal to y.

Thus the fact that the sine of the angle x is equal to y may be expressed in either of the ways

$$\sin x = y, \text{ or } x = \sin^{-1} y.$$

In like manner, the expression $\cos^{-1} y$ signifies the angle whose cosine is equal to y; $\tan^{-1} y$, the angle whose tangent is equal to y, etc.

Note. The student must be careful not to confuse this notation with the *exponent* -1; the -1 power of $\sin x$ is expressed $(\sin x)^{-1}$, and not $\sin^{-1} x$.

77. By aid of the principles of Art. 76, any relation involving direct functions may be transformed into one involving inverse functions.

Take for example the formula

$$\sin(x + y) = \sin x \cos y + \cos x \sin y. \qquad \text{(A)}$$

Let $\quad \sin x = a$, and $\sin y = b$.

Then by Art. 76, $\quad x = \sin^{-1} a$, and $y = \sin^{-1} b$.

Also by Art. 19, $\quad \cos x = \sqrt{1 - \sin^2 x}$,

and $\quad \cos y = \sqrt{1 - \sin^2 y}$.

Putting $\sin x = a$, and $\sin y = b$, we have

$$\cos x = \sqrt{1 - a^2}, \text{ and } \cos y = \sqrt{1 - b^2}.$$

Substituting these values in (A),

$$\sin(\sin^{-1} a + \sin^{-1} b) = a\sqrt{1 - b^2} + b\sqrt{1 - a^2}.$$

Whence by Art. 76,

$$\sin^{-1} a + \sin^{-1} b = \sin^{-1}(a\sqrt{1 - b^2} + b\sqrt{1 - a^2}).$$

EXERCISES.

78. 1. Prove the relation $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$.

By (**25**),
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \quad \text{(Art. 19)}.$$

Dividing each term of the fraction by $\cos^2 x$, we have

$$\cos 2x = \frac{1 - \dfrac{\sin^2 x}{\cos^2 x}}{1 + \dfrac{\sin^2 x}{\cos^2 x}} = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

2. Prove the relation $\dfrac{\sin 5x + \sin x}{\cos 5x + \cos x} = \tan 3x$.

By (**19**) and (**21**),

$$\frac{\sin 5x + \sin x}{\cos 5x + \cos x} = \frac{2\sin \frac{1}{2}(5x + x)\cos\frac{1}{2}(5x - x)}{2\cos\frac{1}{2}(5x + x)\cos\frac{1}{2}(5x - x)}$$
$$= \frac{\sin 3x}{\cos 3x} = \tan 3x.$$

Prove the following relations:

3. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

4. $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot x \cot y - 1}{\cot x \cot y + 1}$.

5. $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1 + \cot x \tan y}{\cot x + \tan y}$.

6. $\sin(45° + y) = \dfrac{\sin y + \cos y}{\sqrt{2}}$.

7. $\tan(60° - y) = \dfrac{\sqrt{3} - \tan y}{1 + \sqrt{3}\tan y}$.

8. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\frac{1}{2}(x+y)$.

9. $\dfrac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x - y)$.

10. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$.

11. $\csc 2x = \frac{1}{2} \sec x \csc x$.

12. $\tan x + \cot x = \dfrac{2}{\sin 2x}$.

13. $\cot x - \tan x = 2 \cot 2x$.

14. $\dfrac{(1 + \tan x)^2 - (1 - \tan x)^2}{(1 + \tan x)^2 + (1 - \tan x)^2} = \sin 2x$.

15. $\sin (x + y) \sin (x - y) = \sin^2 x - \sin^2 y$.

16. $\cos (x + y) \cos (x - y) = \cos^2 x - \sin^2 y$.

17. $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$.

18. $\cos y + \cos (120° + y) + \cos (120° - y) = 0$.

19. $\sin A \sin (B - C) + \sin B \sin (C - A)$
$+ \sin C \sin (A - B) = 0$.

20. $\cos (A + B) \cos (A - B) + \cos (B + C) \cos (B - C)$
$+ \cos (C + A) \cos (C - A) = \cos 2A + \cos 2B + \cos 2C$.

21. $\dfrac{\cos x - \cos 3x}{\sin 3x - \sin x} = \tan 2x$.

22. $\dfrac{\cos 80° + \cos 20°}{\sin 80° - \sin 20°} = \sqrt{3}$.

By putting $x = x + y$ and $y = z$ in (**11**) and (**12**), Art. 65, prove:

23. $\sin (x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z$
$+ \cos x \cos y \sin z - \sin x \sin y \sin z$.

24. $\cos (x + y + z) = \cos x \cos y \cos z - \sin x \sin y \cos z$
$- \sin x \cos y \sin z - \cos x \sin y \sin z$.

By putting $x = 2x$ and $y = x$ in (11), (12), and (15), prove:

25. $\sin 3x = 3\sin x - 4\sin^3 x.$

26. $\cos 3x = 4\cos^3 x - 3\cos x.$

27. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$

Prove the relations:

28. $\sin(2x + y) - 2\sin x \cos(x + y) = \sin y.$

29. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2.$

30. $1 + \cos 2x \cos 2y = 2(\sin^2 x \sin^2 y + \cos^2 x \cos^2 y).$

31. $1 + \tan x \tan 2x = \sec 2x.$

32. $\sin 4x = 4\sin x \cos x - 8\sin^3 x \cos x.$

33. $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1.$

34. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x.$

35. By putting $x = 45°$ and $y = 30°$ in (13) and (14), Art. 66, prove

$$\sin 15° = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}) = \cos 75°,$$
$$\cos 15° = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}) = \sin 75°.$$

36. By putting $x = 30°$ in (34) and (35), Art. 75, prove

$$\tan 15° = 2 - \sqrt{3} = \cot 75°,$$
$$\cot 15° = 2 + \sqrt{3} = \tan 75°.$$

37. By putting $x = 45°$ in (30) and (31), Art. 75, prove

$$\sin 22° 30' = \tfrac{1}{2}\sqrt{2 - \sqrt{2}}, \quad \cos 22° 30' = \tfrac{1}{2}\sqrt{2 + \sqrt{2}}.$$

38. By putting $x = 45°$ in (34) and (35), Art. 75, prove

$$\tan 22° 30' = \sqrt{2} - 1, \quad \cot 22° 30' = \sqrt{2} + 1.$$

VI. LOGARITHMS.

79. Every positive number may be expressed, exactly or approximately, as a power of 10; thus,

$$100 = 10^2;\ 13 = 10^{1.1139\ldots};\ \text{etc.}$$

When thus expressed, the corresponding exponent is called its *Logarithm to the base* 10; thus, 2 is the logarithm of 100 to the base 10, a relation which is written

$$\log_{10} 100 = 2, \text{ or simply } \log 100 = 2.$$

And in general, if $10^x = m$, then $x = \log m$.

80. Any positive number except unity may be taken as the base of a system of logarithms; thus, if $a^x = m$, then $x = \log_a m$.

Logarithms to the base 10 are called *Common Logarithms*, and are the only ones used in numerical computation.

If no base is expressed, the base 10 is understood.

81. We have by Algebra,

$$10^0 = 1, \qquad 10^{-1} = \tfrac{1}{10} \ \ = .1,$$
$$10^1 = 10, \qquad 10^{-2} = \tfrac{1}{100} \ \ = .01,$$
$$10^2 = 100, \qquad 10^{-3} = \tfrac{1}{1000} = .001, \text{ etc.}$$

Whence, by the definition of Art. 79,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$
$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$
$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

Note. The second form of the results for log .1, log .01, etc., is preferable in practice.

82. It is evident from Art. 81 that the logarithm of a number greater than 1 is positive, and that the logarithm of a number between 0 and 1 is negative.

83. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139$.

In this case the characteristic is 1, and the mantissa .1139.

84. It is evident from the first column of Art. 81 that the logarithm of any number between

1 and 10 is equal to 0 plus a decimal;
10 and 100 is equal to 1 plus a decimal;
100 and 1000 is equal to 2 plus a decimal; etc.

Hence, the characteristic of the logarithm of a number with *one* figure to the left of its decimal point, is 0; with *two* figures to the left of the decimal point, is 1; with *three* figures to the left of the decimal point, is 2; etc.

85. In like manner, from the second column of Art. 81, the logarithm of a decimal between

1 and .1 is equal to 9 plus a decimal − 10;
.1 and .01 is equal to 8 plus a decimal − 10;
.01 and .001 is equal to 7 plus a decimal − 10; etc.

Hence, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure, is 9, with −10 after the mantissa; of a decimal with *one* cipher between its point and first figure, is 8, with −10 after the mantissa; of a decimal with *two* ciphers between its point and first figure, is 7, with −10 after the mantissa; etc.

86. For reasons which will be given hereafter, only the mantissa of the logarithm is given in tables of logarithms of numbers; the characteristic must be supplied by the reader.

The rules for characteristic are based on Arts. 84 and 85:

I. *If the number is greater than* 1, *the characteristic is* 1 *less than the number of places to the left of the decimal point.*

II. *If the number is between* 0 *and* 1, *subtract the number of ciphers between the decimal point and first significant figure from* 9, *writing* -10 *after the mantissa.*

Thus, characteristic of $\log 906328.5 = 5$;

characteristic of $\log .007023 = 7$, with -10 after the mantissa.

Note. Some writers, in dealing with the characteristics of negative logarithms, combine the two portions of the characteristic, writing the result as a *negative characteristic* before the mantissa.

Thus, instead of $7.6036 - 10$, the student will frequently find $\overline{3}.6036$, a minus sign being written over the characteristic to denote that it alone is negative, the mantissa being always positive.

PROPERTIES OF LOGARITHMS.

87. *In any system, the logarithm of unity is zero.*

For since $a^0 = 1$, we have $\log_a 1 = 0$ (Art. 79).

88. *In any system, the logarithm of the base itself is unity.*

For since $a^1 = a$, we have $\log_a a = 1$.

89. *In any system whose base is greater than unity, the logarithm of zero is minus infinity.*

For if a is > 1, we have $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$.

Whence by Art. 79, $\log_a 0 = -\infty$.

90. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\} ; \text{ whence by Art. 79, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Multiplying, we have

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $\log_a mn = x + y.$

Substituting the values of x and y, we have

$$\log_a mn = \log_a m + \log_a n.$$

In like manner, the theorem may be proved for the product of three or more factors.

91. By aid of the theorem of Art. 90, the logarithm of any composite number may be found when the logarithms of its factors are known.

1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned} \log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \\ &= 3 \times \log 2 + 2 \times \log 3 \\ &= .9030 + .9542 = 1.8572. \end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$; find the values of the following:

2. $\log 6$.	**7**. $\log 21$.	**12**. $\log 98$.	**17**. $\log 135$.
3. $\log 14$.	**8**. $\log 63$.	**13**. $\log 105$.	**18**. $\log 168$.
4. $\log 8$.	**9**. $\log 56$.	**14**. $\log 112$.	**19**. $\log 147$.
5. $\log 12$.	**10**. $\log 84$.	**15**. $\log 144$.	**20**. $\log 375$.
6. $\log 15$.	**11**. $\log 45$.	**16**. $\log 216$.	**21**. $\log 343$.

92. *In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{aligned} a^x &= m \\ a^y &= n \end{aligned}\right\}; \text{ whence, } \left\{\begin{aligned} x &= \log_a m, \\ y &= \log_a n. \end{aligned}\right.$$

Dividing, we have $\frac{a^x}{a^y}=\frac{m}{n}$, or $a^{x-y}=\frac{m}{n}$.

Whence, $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$.

93. 1. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - .3010 = .6990.$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

2. $\log \frac{7}{3}$.

3. $\log \frac{10}{7}$.

4. $\log 3\frac{1}{3}$.

5. $\log 35$.

6. $\log \frac{21}{16}$.

7. $\log 125$.

8. $\log \frac{42}{25}$.

9. $\log 175$.

10. $\log 11\frac{1}{9}$.

11. $\log 7\frac{1}{7}$.

12. $\log \frac{35}{6}$.

13. $\log 5\frac{4}{9}$.

94. *In any system, the logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation

$$a^x = m;\ \text{whence, } x = \log_a m.$$

Raising both members to the pth power, we have

$$a^{px} = m^p;\ \text{whence, } \log_a m^p = px = p \log_a m.$$

95. *In any system, the logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For, $\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$ (Art. 94).

96. 1. Given $\log 2 = .3010$; find the logarithm of $2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017.$$

Note. To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find the logarithm of $\sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596.$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

3. $\log 3^{\frac{3}{5}}$.	**7.** $\log 12^{\frac{2}{3}}$.	**11.** $\log 15^{\frac{5}{6}}$.	**15.** $\log \sqrt[6]{5}$.
4. $\log 2^9$.	**8.** $\log 21^{\frac{1}{2}}$.	**12.** $\log \sqrt{7}$.	**16.** $\log \sqrt[4]{35}$.
5. $\log 7^5$.	**9.** $\log 14^4$.	**13.** $\log \sqrt[3]{3}$.	**17.** $\log \sqrt[9]{98}$.
6. $\log 5^{\frac{1}{5}}$.	**10.** $\log 25^{\frac{7}{3}}$.	**14.** $\log \sqrt[7]{2}$.	**18.** $\log \sqrt[12]{126}$.

19. Find the logarithm of $(2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By Art. 90, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$$= \tfrac{1}{3} \log 2 + \tfrac{5}{4} \log 3$$
$$= .1003 + .5964 = .6967.$$

Find the values of the following:

20. $\log \left(\frac{10}{3}\right)^5$.	**22.** $\log (3^{\frac{1}{6}} \times 2^{\frac{3}{5}})$.	**24.** $\log \sqrt{\frac{7}{3}}$.	**26.** $\log \sqrt[3]{\frac{28}{5}}$.
21. $\log \frac{7^{\frac{3}{4}}}{5^{\frac{2}{3}}}$.	**23.** $\log 3\sqrt[4]{7}$.	**25.** $\log \frac{\sqrt[3]{7}}{\sqrt[5]{2}}$.	**27.** $\log \frac{\sqrt{42}}{10^{\frac{2}{3}}}$.

97. *In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.*

To illustrate, suppose that $\log 3.053 = .4847$.

Then,

$$\begin{aligned}
\log 30.53 &= \log (10 \times 3.053) &&= \log 10 + \log 3.053 \\
&= 1 + .4847 &&= 1.4847; \\
\log 305.3 &= \log (100 \times 3.053) &&= \log 100 + \log 3.053 \\
&= 2 + .4847 &&= 2.4847; \\
\log .03053 &= \log (.01 \times 3.053) &&= \log .01 + \log 3.053 \\
&= 8 - 10 + .4847 &&= 8.4847 - 10; \text{ etc.}
\end{aligned}$$

It is evident from the foregoing that if a number is multiplied or divided by any integral power of 10, thus producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

Thus, if $\log 3.053 = .4847$, then

$$\log 30.53 = 1.4847, \qquad \log .3053 = 9.4847 - 10,$$
$$\log 305.3 = 2.4847, \qquad \log .03053 = 8.4847 - 10,$$
$$\log 3053. = 3.4847, \qquad \log .003053 = 7.4847 - 10, \text{ etc.}$$

Note. The reason will now be seen for the statement made in Art. 86, that only the mantissæ are given in a table of logarithms of numbers. For, to find the logarithm of any number, we have only to take from the table the mantissa corresponding to its sequence of figures, and the characteristic may then be prefixed in accordance with the rules of Art. 86.

This property of logarithms is only enjoyed by the common system, and constitutes its superiority over others for the purposes of numerical computation.

98. 1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

$$\log 432 = \log (2^4 \times 3^3) = 4 \log 2 + 3 \log 3$$
$$= 1.2040 + 1.4313 = 2.6353.$$

Then by Art. 97, the *mantissa* of the result is .6353.

Whence by Art. 86, $\log .00432 = 7.6353 - 10.$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$; find the values of the following:

2. $\log 1.8$.

3. $\log 2.25$.

4. $\log .196$.

5. $\log .048$.

6. $\log 38.4$.

7. $\log .0054$.

8. $\log .000315$.

9. $\log 7350$.

10. $\log 4.05$.

11. $\log .448$.

12. $\log 302.4$.

13. $\log .06174$.

14. $\log (8.1)^7$.

15. $\log \sqrt[5]{9.6}$.

16. $\log (22.4)^{\frac{1}{3}}$.

99. *To prove the relation*

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ b^y = m \end{array}\right\} ; \text{ whence, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_b m. \end{array}\right.$$

From the assumed equations, we have

$$a^x = b^y, \text{ or } a^{\frac{x}{y}} = b.$$

Whence, $\log_a b = \frac{x}{y}$, or $y = \frac{x}{\log_a b}$.

Substituting the values of x and y,

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

By aid of this relation, if the logarithm of a quantity m to a certain base a is known, its logarithm to any other base b may be found by dividing by the logarithm of b to the base a.

100. *To prove the relation*

$$\log_b a \times \log_a b = 1.$$

Putting $m = a$ in the result of Art. 99, we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \quad \text{(Art. 88)}.$$

Whence, $\log_b a \times \log_a b = 1.$

APPLICATIONS.

101. The value of an arithmetical quantity, in which the operations indicated involve only multiplication, division, involution, or evolution, may be most conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

In operations with negative characteristics the rules of Algebra must be followed.

102. 1. Find the value of $.0631 \times 7.208 \times .51272$.

By Art. 90, $\log(.0631 \times 7.208 \times .51272)$

$$= \log .0631 + \log 7.208 + \log .51272.$$

$$\begin{aligned} \log .0631 &= 8.8000 - 10 \\ \log 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \end{aligned}$$

Adding, $\therefore$ log of result $= 19.3677 - 20$

$= 9.3677 - 10$ (see Note 1)

Number corresponding to $9.3677 - 10 = .23317$.

Note 1. If the sum is a negative logarithm, it should be reduced so that the negative portion of the characteristic may be -10.

Thus, $19.3677 - 20$ is reduced to $9.3677 - 10$.

2. Find the value of $\dfrac{336.8}{7984}$.

By Art. 92, $\log \dfrac{336.8}{7984} = \log 336.8 - \log 7984.$

$$\begin{aligned} \log 336.8 &= 12.5273 - 10 \quad \text{(see Note 2)} \\ \log 7984 &= \underline{3.9022} \end{aligned}$$

Subtracting, $\therefore$ log of result $= 8.6251 - 10$

Number corresponding $= .04218$.

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By Art. 94, $\log(.07396)^5 = 5 \times \log .07396$.

$$\begin{aligned} \log .07396 &= 8.8690 - 10 \\ &\quad\underline{5} \\ &\quad 44.3450 - 50 \\ &= 4.3450 - 10 \quad \text{(see Note 1)} \\ &= \log .000002213. \end{aligned}$$

4. Find the value of $\sqrt[3]{.035063}$.

By Art. 95, $\log\sqrt[3]{.035063} = \frac{1}{3}\log .035063.$

$$
\begin{array}{rl}
\log .035063 = & 8.5449 - 10 \\
& \underline{20. \qquad - 20} \text{ (see Note 3)} \\
& 3\overline{)28.5449 - 30} \\
& \quad 9.5150 - 10 = \log .3274.
\end{array}
$$

Note 3. To divide a negative logarithm, add to both parts such a multiple of 10 as will make the negative portion of the characteristic exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, add 20 to both parts of the logarithm, giving the result $28.5449 - 30$. Dividing this by 3, the quotient is $9.5150 - 10$.

ARITHMETICAL COMPLEMENT.

103. The *Arithmetical Complement* of the logarithm of a number, or briefly the *Cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus, $\quad \text{colog } 409 = \log\frac{1}{409} = \log 1 - \log 409.$

$$
\begin{array}{rl}
\log 1 = & 10. \qquad - 10 \quad \text{(Note 2, Art. 102)} \\
\log 409 = & \underline{\;2.6117 \qquad\quad} \\
\therefore \text{ colog } 409 = & 7.3883 - 10.
\end{array}
$$

Again, $\quad \text{colog } .067 = \log\frac{1}{.067} = \log 1 - \log .067.$

$$
\begin{array}{rl}
\log 1 = & 10. \qquad - 10 \\
\log .067 = & \underline{8.8261 - 10} \\
\therefore \text{ colog } .067 = & 1.1739.
\end{array}
$$

The following rule is evident from the above:

To find the cologarithm of a number, subtract its logarithm from $10 - 10$.

Note. The cologarithm may be obtained from the logarithm by subtracting the last *significant* figure from 10 and each of the others from 9, -10 being written after the result in the case of a positive logarithm.

104. *Example.* Find the value of $\dfrac{.51384}{8.709 \times .0946}$.

$$\log\frac{.51384}{8.709 \times .0946} = \log\left(.51384 \times \frac{1}{8.709} \times \frac{1}{.0946}\right)$$

$$= \log .51384 + \log\frac{1}{8.709} + \log\frac{1}{.0946}$$

$$= \log .51384 + \text{colog}\, 8.709 + \text{colog}\, .0946.$$

$$\begin{aligned} \log .51384 &= 9.7109 - 10 \\ \text{colog}\, 8.709 &= 9.0601 - 10 \\ \text{colog}\, .0946 &= \underline{1.0241 \qquad\quad} \\ & 9.7951 - 10 = \log .6239. \end{aligned}$$

It is evident from the above that the logarithm of a fraction is equal to the logarithm of the numerator *plus* the cologarithm of the denominator.

Or in general, to find the logarithm of a fraction whose terms are composed of factors,

Add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.

Note. The value of the above fraction may be found without using cologarithms, by the following formula:

$$\log\frac{.51384}{8.709 \times .0946} = \log .51384 - \log(8.709 \times .0946)$$

$$= \log .51384 - (\log 8.709 + \log .0946).$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

EXAMPLES.

105. Note. A *negative* quantity can have no common logarithm, as is evident from the definition of Art. 79. If negative quantities occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3, p. 65, the value of $721.3 \times (-3.0528)$ may be obtained by finding the value of 721.3×3.0528, and putting a negative sign before the result. See also Ex. 34, p. 66.

Find by logarithms the values of the following:

1. $9.238 \times .9152$.

2. $130.36 \times .08237$.

3. $721.3 \times (-3.0528)$.

4. $(-4.3264) \times (-.050377)$.

5. $.27031 \times .042809$.

6. $(-.063165) \times 11.134$.

7. $\dfrac{401.8}{52.37}$.

8. $\dfrac{7.2321}{10.813}$.

9. $\dfrac{-.3384}{.08659}$.

10. $\dfrac{9.163}{.0051422}$.

11. $\dfrac{22518}{64327}$.

12. $\dfrac{.007514}{-.015822}$.

13. $\dfrac{3.3681}{12.853 \times .6349}$.

14. $\dfrac{15.008 \times (-.0843)}{.06376 \times 4.248}$.

15. $\dfrac{(-2563) \times .03442}{714.8 \times (-.511)}$.

16. $\dfrac{121.6 \times (-9.025)}{(-48.3) \times 3662 \times (-.0856)}$.

17. $(23.86)^3$.

18. $(.532)^8$.

19. $(-1.0246)^7$.

20. $(.09323)^5$.

21. $5^{\frac{2}{3}}$.

22. $(.8)^{\frac{2}{5}}$.

23. $(-3.16)^{\frac{4}{3}}$.

24. $(.021)^{\frac{5}{2}}$.

25. $\sqrt{2}$.

26. $\sqrt[4]{5}$.

27. $\sqrt[5]{-3}$.

28. $\sqrt{.4294}$.

29. $\sqrt[3]{.02305}$.

30. $\sqrt[8]{1000}$.

31. $\sqrt[7]{-.00951}$.

32. $\sqrt[5]{.0001011}$.

33. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log \sqrt[3]{5} + \text{colog}\, 3^{\frac{5}{6}}$$

$$= \log 2 + \tfrac{1}{3} \log 5 + \tfrac{5}{6} \text{colog}\, 3.$$

$$\begin{array}{lr} & \log 2 = .3010 \\ \log 5 = .6990\,; & \text{divide by } 3 = .2330 \\ \text{colog}\, 3 = 9.5229 - 10\,; & \text{multiply by } \tfrac{5}{6} = \underline{9.6024 - 10} \\ & .1364 = \log 1.3691. \end{array}$$

34. Find the value of $\sqrt[3]{\dfrac{-.03296}{7.962}}$.

$$\log \sqrt[3]{\frac{.03296}{7.962}} = \tfrac{1}{3}\log\frac{.03296}{7.962} = \tfrac{1}{3}(\log .03296 - \log 7.962).$$

$$\begin{aligned} \log .03296 &= 8.5180 - 10 \\ \log 7.962 &= 0.9010 \\ &\ 3)\overline{27.6170 - 30} \\ &\ 9.2057 - 10 = \log .16059. \end{aligned}$$

Result, $-.16059$.

Find the values of the following:

35. $2^{\frac{3}{2}} \times 3^{\frac{2}{3}}$.

36. $\dfrac{3^{\frac{5}{8}}}{4^{\frac{2}{3}}}$.

37. $\dfrac{5^{\frac{3}{7}}}{(-10)^{\frac{2}{9}}}$.

38. $\left(\dfrac{6}{7}\right)^{\frac{5}{2}}$.

39. $\left(\dfrac{35}{113}\right)^{\frac{3}{8}}$.

40. $\left(\dfrac{.08726}{.1321}\right)^{\frac{5}{3}}$.

41. $\sqrt[8]{\dfrac{21}{13}}$.

42. $\sqrt[9]{-\dfrac{3}{7}}$.

43. $\sqrt[5]{\dfrac{2}{3}} \div \sqrt[3]{\dfrac{3}{5}}$.

44. $\sqrt[8]{2} \times \sqrt[5]{3} \times \sqrt[7]{.01}$.

45. $\sqrt[5]{\dfrac{3258}{49309}}$.

46. $\left(\dfrac{-31.63}{429}\right)^{\frac{3}{17}}$.

47. $\dfrac{100^{\frac{2}{3}}}{(.7325)^{\frac{3}{7}}}$.

48. $\dfrac{\sqrt[3]{.0001289}}{\sqrt[4]{.0008276}}$.

49. $\dfrac{(-.7469)^{\frac{5}{3}}}{-(.2345)^{\frac{7}{2}}}$.

50. $\dfrac{\sqrt[11]{.0073}}{(.68291)^{\frac{5}{2}}}$.

51. $\dfrac{\sqrt{5.955} \times \sqrt[3]{61.2}}{\sqrt[5]{298.54}}$.

52. $(538.2 \times .0005969)^{\frac{1}{8}}$.

53. $(18.9503)^{11} \times (-.1)^{14}$.

54. $\sqrt[6]{3734.9 \times .00001108}$.

55. $(2.6317)^{\frac{3}{4}} \times (.71272)^{\frac{2}{5}}$.

56. $\dfrac{\sqrt[3]{-.008193} \times (.06285)^{\frac{3}{2}}}{-.98342}$.

57. $\sqrt{.035} \times \sqrt[6]{.62667} \times \sqrt[3]{.0072103}$.

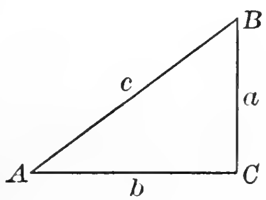
VII. SOLUTION OF RIGHT TRIANGLES.

106. The six *elements* of a triangle are its three sides and its three angles.

We know by Geometry that, in general, a triangle is completely determined when three of its elements are known, provided one of them is a side. The *solution* of a triangle is the process of computing the unknown from the given elements.

107. To solve a *right triangle*, *two* elements must be given in addition to the right angle, one of which must be a side.

The various cases which can occur may all be solved by aid of the following formulæ:



$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b},$$

$$\sin B = \frac{b}{c}, \qquad \cos B = \frac{a}{c}, \qquad \tan B = \frac{b}{a}.$$

Case I. *When the given elements are a side and an angle.*

The proper formula for computing either of the remaining sides may be found by the following rule:

Take that function of the angle which involves the given side and the required side.

Case II. *When both the given elements are sides.*

First calculate one of the angles by aid of either of the formulæ involving the given elements, and then compute the remaining side as in Case I.

EXAMPLES.

108. **1.** Given $c = 203.76$, $B = 21°\ 43'$. Find a and b.

In this case the formulæ to be used are

$$\cos B = \frac{a}{c}, \quad \text{and} \quad \sin B = \frac{b}{c}.$$

Whence, $a = c \cos B$, and $b = c \sin B$.

By logarithms, $\log a = \log c + \log \cos B$,

and $\log b = \log c + \log \sin B$.

$$\begin{array}{rl} \log c = & 2.3091 \\ \log \cos B = & \underline{9.9681 - 10} \\ \log a = & 2.2772 \\ \therefore\ a = & 189.3. \end{array} \qquad \begin{array}{rl} \log c = & 2.3091 \\ \log \sin B = & \underline{9.5682 - 10} \\ \log b = & 1.8773 \\ \therefore\ b = & 75.38. \end{array}$$

2. Given $a = 13$, $A = 67°\ 7'$. Find b and c.

In this case, $\tan A = \frac{a}{b}$, and $\sin A = \frac{a}{c}$.

Whence, $b = \dfrac{a}{\tan A}$, and $c = \dfrac{a}{\sin A}$.

By logarithms, $\log b = \log a - \log \tan A$,

and $\log c = \log a - \log \sin A$.

$$\begin{array}{rl} \log a = & 1.1139 \\ \log \tan A = & \underline{0.3746} \\ \log b = & 0.7393 \\ \therefore\ b = & 5.486. \end{array} \qquad \begin{array}{rl} \log a = & 1.1139 \\ \log \sin A = & \underline{9.9644 - 10} \\ \log c = & 1.1495 \\ \therefore\ c = & 14.11. \end{array}$$

3. Given $b = .1512$, $c = .3081$. Find A and a.

In this case, $\cos A = \frac{b}{c}$, and $\tan A = \frac{a}{b}$, or $a = b \tan A$.

Whence, $\log \cos A = \log b - \log c$,

and $\log a = \log b + \log \tan A$.

$$\begin{aligned} \log b &= 9.1796 \\ \log c &= \underline{9.4887} \\ \log \cos A &= 9.6909 \\ \therefore\ A &= 60^\circ\ 36.4'. \end{aligned} \qquad \begin{aligned} \log b &= 9.1796 \\ \log \tan A &= \underline{0.2493} \\ \log a &= 9.4289 \\ \therefore\ a &= \ .2685. \end{aligned}$$

Note. It is customary in practice to omit writing the -10 after the mantissa of a negative logarithm, as illustrated in Ex. 3.

109. In the Trigonometrical solution of a triangle by the method of Case II., it is necessary to first find one of the angles, and the remaining side may then be calculated.

It is possible however to obtain the third side directly, without first finding the angle, by Geometrical methods.

Thus in Ex. 3, Art. 108, we have by Geometry,

$$a^2 + b^2 = c^2.$$

Whence, $$a = \sqrt{c^2 - b^2} = \sqrt{(c+b)\,(c-b)}.$$

By logarithms, $$\log a = \tfrac{1}{2}\,[\log\,(c+b) + \log\,(c-b)].$$

$$\begin{aligned} c + b = .4593\ ;\ \log &= 9.6621 - 10 \\ c - b = .1569\ ;\ \log &= \underline{9.1956 - 10} \\ 2\,&\overline{)18.8577 - 20} \\ \log a &= 9.4289 - 10 \\ \therefore\ a &= \ .2685, \text{ as before.} \end{aligned}$$

If the given sides are a and b, the formula for c is $\sqrt{a^2 + b^2}$, which is not adapted to logarithmic computation.

In such a case it is usually shorter to proceed according to the rule of Art. 107.

EXAMPLES.

110. Solve the following right triangles:

1. Given $A = 43^\circ\ 30'$, $c = 11.2$.
2. Given $B = 68^\circ\ 50'$, $a = 729.3$.
3. Given $B = 62^\circ\ 56'$, $b = 47.7$.
4. Given $a = .624$, $c = .91$.

5. Given $a = 5$, $b = 2$.

6. Given $A = 72° 7'$, $a = 83.4$.

7. Given $B = 32° 10'$, $c = .02728$.

8. Given $b = 2.887$, $c = 5.11$.

9. Given $A = 52° 41'$, $b = 4247$.

10. Given $a = 101$, $b = 116$.

11. Given $A = 43° 22'$, $a = 158.3$.

12. Given $A = 58° 39'$, $c = 35.73$.

13. Given $a = 204.2$, $c = 275.3$.

14. Given $B = 30°$, $b = 1.6438$.

15. Given $A = 22° 14'$, $b = 13.242$.

16. Given $B = 10° 51'$, $c = .7264$.

17. Given $a = 638.5$, $b = 501.2$.

18. Given $A = 78° 17'$, $a = 203.8$.

19. Given $b = .02497$, $c = .04792$.

20. Given $B = 2° 19' 30''$, $a = 1875.3$.

21. Given $a = 24.67$, $b = 33.02$.

22. Given $b = 1.4367$, $c = 3.4653$.

23. Given $B = 6° 12.3'$, $c = 37206$.

24. Given $A = 64° 1.3'$, $b = 200.05$.

25. Given $a = 340.06$, $b = 231.69$.

26. Given $a = 1.7087$, $c = 2.0008$.

27. Given $B = 21° 33' 51''$, $a = .82109$.

28. Given $A = 74° 0' 18''$, $c = 275.62$.

29. Given $B = 34° 14' 37''$, $b = 120.22$.

30. Given $a = 10.107$, $b = 17.303$.

Solve the following isosceles triangles, in which A and B are the equal angles, and a, b, and c denote the sides opposite the angles A, B, and C, respectively:

31. Given $A = 68° 57'$, $b = 35.091$.

32. Given $B = 27° 8'$, $c = 3.088$.

33. Given $C = 80° 47'$, $b = 2103.2$.

34. Given $a = 79.24$, $c = 106.62$.

35. Given $A = 70° 19'$, $c = .5623$.

36. Given $C = 151° 28'$, $c = 95.47$.

37. A regular pentagon is inscribed in a circle whose diameter is 24 inches. Find the length of its side.

38. At a distance of 100 feet from the base of a tower, the angle of elevation of its top is observed to be 38°. Find its height.

39. What is the angle of elevation of the sun when a tower whose height is 103.7 feet casts a shadow 167.3 feet in length?

40. If the diameter of a circle is 3268, find the angle at the centre subtended by an arc whose chord is 1027.

41. If the diameter of the earth is 7912 miles, what is the distance of the remotest point of the surface visible from the summit of a mountain $1\frac{1}{4}$ miles in height?

42. Find the length of the diagonal of a regular pentagon whose side is 7.028 inches.

43. What is the angle of elevation of a mountain-slope which rises 238 feet in a horizontal distance of one-eighth of a mile?

44. From the top of a lighthouse, 133 feet above the sea, the angle of depression of a buoy is observed to be 18° 25'. Required the horizontal distance of the buoy.

45. A ship is sailing due east at the rate of 7.8 miles an hour. A headland is observed to bear due north at 10.37 A.M., and 33° west of north at 12.43 P.M. Find the distance of the headland from each point of observation.

46. If a chord of 41.36 feet subtends an arc of 145° 37′, what is the radius of the circle?

47. The length of the side of a regular octagon is 12 inches. Find the radii of the inscribed and circumscribed circles.

48. How far from the foot of a pole 80 feet high must an observer stand, so that the angle of elevation of the top of the pole may be 10°?

49. If the diagonal of a regular pentagon is 32.83 inches, what is the radius of the circumscribed circle?

50. From the top of a tower, the angle of depression of the extremity of a horizontal base line, 1000 feet in length measured from the foot of the tower, is observed to be 21° 16′ 37″. Find the height of the tower.

51. If the radius of a circle is 723.29, what is the length of the chord which subtends an arc of 35° 13′?

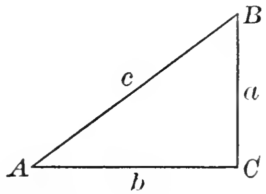
52. A regular hexagon is circumscribed about a circle whose diameter is 10 inches. Find the length of its side.

53. From the top of a lighthouse, 200 feet above the sea, the angles of depression of two boats in line with the lighthouse are observed to be 14° and 32° respectively. What is the distance between the boats?

54. A ship is sailing due east at a uniform rate of speed. At 7 A.M., a lighthouse is observed bearing due north, 10.32 miles distant, and at 7.30 A.M. it bears 18° 13′ west of north. Find the rate of sailing of the ship and the bearing of the lighthouse at 10 A.M.

FORMULÆ FOR THE AREA OF A RIGHT TRIANGLE.

111. CASE I. *Given the hypotenuse and an acute angle.*



Denoting the area by K, we have by Geometry,

$$2K = ab.$$

But by Art. 13, $a = c \sin A$,

and $b = c \cos A$.

Whence, $2K = c^2 \sin A \cos A = \frac{1}{2} c^2 \sin 2A$ (Art. 74).

That is, $$4K = c^2 \sin 2A. \quad (36)$$

In like manner,

$$4K = c^2 \sin 2B. \quad (37)$$

CASE II. *Given an angle and its opposite side.*

We have, $\cot A = \frac{b}{a}$, or $b = a \cot A$.

Whence, $$2K = a \cdot a \cot A = a^2 \cot A. \quad (38)$$

In like manner,

$$2K = b^2 \cot B. \quad (39)$$

CASE III. *Given an angle and its adjacent side.*

We have, $\cot A = \tan B$ (Art. 14).

Whence by (38),

$$2K = a^2 \tan B. \quad (40)$$

In like manner,

$$2K = b^2 \tan A. \quad (41)$$

Case IV. *Given the hypotenuse and another side.*

Since $a^2+b^2=c^2$, we have

$$2K=ab=a\sqrt{c^2-a^2}$$
$$=a\sqrt{(c+a)(c-a)}. \quad (42)$$

In like manner,

$$2K=b\sqrt{(c+b)(c-b)}. \quad (43)$$

Case V. *Given the two sides about the right angle.*

In this case, $2K=ab.$ **(44)**

EXAMPLES.

112. 1. Given $c=10.36$, $B=75°$; find the area.

By (37), $4K=c^2\sin 2B.$

Whence, $\log(4K)=2\log c+\log\sin 2B.$

$\log c = 1.0153$; multiply by $2 = 2.0306$

$2B=150°$; $\log\sin = 9.6990-10$

$\log(4K)=1.7296$

$\therefore 4K=53.65$, and $K=13.41$.

Note. To find $\log\sin 150°$, take either $\log\cos 60°$ or $\log\sin 30°$. (See page 10 of the explanation of the tables.)

Find the areas of the following triangles:

2. Given $A=19°\ 36'$, $a=22.17$.

3. Given $B=24°\ 7'$, $a=.8213$.

4. Given $a=149.31$, $b=76.29$.

5. Given $b=.3056$, $c=.6601$.

6. Given $A=30°\ 56'\ 20''$, $c=192.9$.

7. Given $A=58°\ 52'$, $b=.05207$.

8. Given $a=.932$, $c=2.786$.

9. Given $B=72°\ 25'$, $c=27.283$.

10. Given $B=29°\ 18'\ 15''$, $b=.33784$.

VIII. GENERAL PROPERTIES OF TRIANGLES.

113. *In any triangle, the sides are proportional to the sines of their opposite angles.*

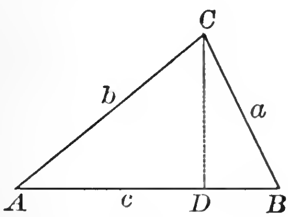


FIG. 1.

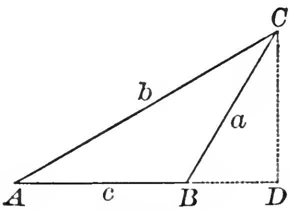


FIG. 2.

There will be two cases, according as the angles are all acute (Fig. 1), or one of them obtuse (Fig. 2).

In each case let CD be drawn perpendicular to AB.

Then in either figure, we have

$$\sin A = \frac{CD}{b}.$$

In Fig. 1, $$\sin B = \frac{CD}{a},$$

and in Fig. 2, $$\begin{aligned}\sin B &= \sin(180° - CBD)\\ &= \sin CBD \quad \text{(Art. 47)}\\ &= \frac{CD}{a}.\end{aligned}$$

Dividing these equations, we have in either case

$$\frac{\sin A}{\sin B} = \frac{\frac{CD}{b}}{\frac{CD}{a}} = \frac{a}{b}. \tag{45}$$

In like manner, we may prove

$$\frac{\sin B}{\sin C} = \frac{b}{c}, \tag{46}$$

and

$$\frac{\sin C}{\sin A} = \frac{c}{a}. \tag{47}$$

The above results may be expressed more compactly as follows:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

114. *In any triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Formula (45) of Art. 113 may be put in the form

$$a : b = \sin A : \sin B.$$

Whence, by composition and division,

$$a + b : a - b = \sin A + \sin B : \sin A - \sin B,$$

or,

$$\frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

But by Art. 73,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$$

Whence,

$$\frac{a + b}{a - b} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}. \tag{48}$$

In like manner, we may prove

$$\frac{b + c}{b - c} = \frac{\tan \frac{1}{2}(B + C)}{\tan \frac{1}{2}(B - C)}, \tag{49}$$

and

$$\frac{c + a}{c - a} = \frac{\tan \frac{1}{2}(C + A)}{\tan \frac{1}{2}(C - A)}. \tag{50}$$

115. Since $A+B=180^\circ - C$, we have

$$\tan \tfrac{1}{2}(A+B) = \tan(90^\circ - \tfrac{1}{2}C) = \cot \tfrac{1}{2}C \quad \text{(Art. 14)}.$$

Thus formula (**48**) may be put in the form

$$\frac{a+b}{a-b} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)}. \tag{51}$$

116. *In any triangle, the square of any side is equal to the sum of the squares of the other two sides, minus twice their product into the cosine of their included angle.*

Case I. *When the included angle A is acute.*

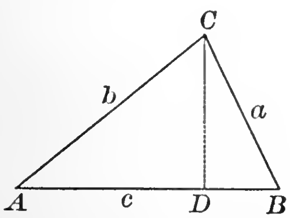


Fig. 1.

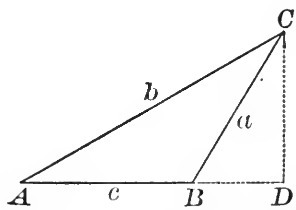


Fig. 2.

There will be two cases, according as the remaining angles are both acute (Fig. 1), or one of them obtuse (Fig. 2).

In each case let CD be drawn perpendicular to AB.

Then in Fig. 1, $\quad BD = c - AD,$

and in Fig. 2, $\quad BD = AD - c.$

Squaring, we have in either case,

$$\overline{BD}^2 = \overline{AD}^2 + c^2 - 2c \times AD.$$

Adding $\overline{CD}^2$ to both members,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{CD}^2 + c^2 - 2c \times AD.$$

But, $\quad \overline{BD}^2 + \overline{CD}^2 = a^2$, and $\overline{AD}^2 + \overline{CD}^2 = b^2$.

Also, $\quad \cos A = \dfrac{AD}{b}$, or $AD = b \cos A$.

Therefore, $$a^2 = b^2 + c^2 - 2bc \cos A. \tag{52}$$

CASE II. *When the included angle A is obtuse.*

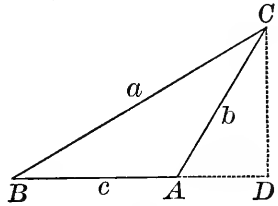


FIG. 3.

In Fig. 3, $BD = AD + c.$

Squaring, and adding $\overline{CD}^2$ to both members,

$$\overline{BD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{CD}^2 + c^2 + 2\,c \times AD.$$

But, $\overline{BD}^2 + \overline{CD}^2 = a^2$, and $\overline{AD}^2 + \overline{CD}^2 = b^2$.

Also,
$$\cos A = \cos(180° - CAD)$$
$$= -\cos CAD \quad \text{(Art. 47)}$$
$$= -\frac{AD}{b}.$$

Whence, $AD = -b\cos A.$

Therefore, $a^2 = b^2 + c^2 - 2\,bc\cos A.$

In like manner, we may prove

$$b^2 = c^2 + a^2 - 2\,ca\cos B, \tag{53}$$

and

$$c^2 = a^2 + b^2 - 2\,ab\cos C. \tag{54}$$

117. *To express the cosines of the angles of a triangle in terms of the sides of the triangle.*

From (52), Art. 116,

$$a^2 = b^2 + c^2 - 2\,bc\cos A.$$

Whence,

$$2\,bc\cos A = b^2 + c^2 - a^2,$$

or,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{55}$$

In like manner, we have

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca}, \tag{56}$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}. \tag{57}$$

118. *To express the sines, cosines, and tangents of the half-angles of a triangle in terms of the sides of the triangle.*

From (**55**), Art. 117,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

Subtracting both members from unity,

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{a^2 - b^2 + 2\,bc - c^2}{2\,bc}.$$

Whence by (**A**), Art. 75,

$$2 \sin^2 \tfrac{1}{2} A = \frac{a^2 - (b - c)^2}{2\,bc},$$

or,

$$\sin^2 \tfrac{1}{2} A = \frac{(a - b + c)\,(a + b - c)}{4\,bc}.$$

Denoting $a + b + c$ by $2\,s$, so that s is the half-sum of the sides of the triangle, we have

$$a - b + c = (a + b + c) - 2\,b = 2\,s - 2\,b = 2\,(s - b),$$

and

$$a + b - c = (a + b + c) - 2\,c = 2\,s - 2\,c = 2\,(s - c).$$

Hence,

$$\sin^2 \tfrac{1}{2} A = \frac{4\,(s - b)\,(s - c)}{4\,bc},$$

or,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)\,(s - c)}{bc}}. \tag{58}$$

In like manner, we may prove

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s - c)\,(s - a)}{ca}}, \tag{59}$$

and

$$\sin \tfrac{1}{2} C = \sqrt{\frac{(s - a)\,(s - b}{ab}}. \tag{60}$$

Again, adding both members of (**55**) to unity, we have

$$1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}=\frac{b^2+2bc+c^2-a^2}{2bc}.$$

Whence by (A), Art. 75,

$$2\cos^2\tfrac{1}{2}A=\frac{(b+c)^2-a^2}{2bc},$$

or,
$$\cos^2\tfrac{1}{2}A=\frac{(b+c+a)\,(b+c-a)}{4bc}.$$

But, $b+c+a=2s,$

and $b+c-a=(b+c+a)-2a=2\,(s-a).$

Hence,
$$\cos^2\tfrac{1}{2}A=\frac{4s\,(s-a)}{4bc},$$

or,
$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}. \qquad (\mathbf{61})$$

In like manner,

$$\cos\tfrac{1}{2}B=\sqrt{\frac{s(s-b)}{ca}}, \qquad (\mathbf{62})$$

and
$$\cos\tfrac{1}{2}C=\sqrt{\frac{s(s-c)}{ab}}. \qquad (\mathbf{63})$$

Dividing (**58**) by (**61**), we obtain

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)\,(s-c)}{bc}}\sqrt{\frac{bc}{s(s-a)}}$$

$$=\sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}}. \qquad (\mathbf{64})$$

In like manner,

$$\tan\tfrac{1}{2}B=\sqrt{\frac{(s-c)\,(s-a)}{s(s-b)}}, \qquad (\mathbf{65})$$

and
$$\tan\tfrac{1}{2}C=\sqrt{\frac{(s-a)\,(s-b)}{s(s-c)}}. \qquad (\mathbf{66})$$

Note. Since each angle of a triangle is less than 180°, its half is less than 90°; hence the *positive sign* must be taken before the radical in each of the formulæ of Art. 118.

AREA OF AN OBLIQUE TRIANGLE.

119. CASE I. *Given two sides and their included angle.*

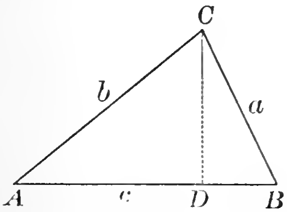


FIG. 1.

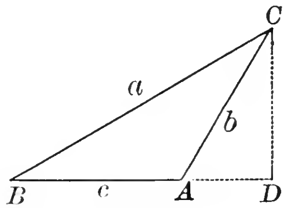


FIG. 2.

There will be two cases, according as the included angle A is acute (Fig. 1), or obtuse (Fig. 2).

In each case let CD be drawn perpendicular to AB.

Then, denoting the area of the triangle by K, we have by Geometry,

$$2K = c \times CD.$$

But in Fig. 1, $\sin A = \dfrac{CD}{b}$,

and in Fig. 2, $\sin A = \sin(180° - CAD)$

$$= \sin CAD \text{ (Art. 47)} = \frac{CD}{b}.$$

Whence, in either figure,

$$CD = b \sin A.$$

Therefore, $$2K = bc \sin A. \qquad (67)$$

In like manner,

$$2K = ca \sin B, \qquad (68)$$

and $$2K = ab \sin C. \qquad (69)$$

CASE II. *Given a side and all the angles.*

By (69), $$2K = ab \sin C.$$

But by Art. 113,

$$\frac{b}{a} = \frac{\sin B}{\sin A}, \quad \text{or} \quad b = \frac{a \sin B}{\sin A}.$$

Substituting, $2K = a \times \dfrac{a \sin B}{\sin A} \times \sin C$

$$= \frac{a^2 \sin B \sin C}{\sin A}. \tag{70}$$

In like manner,

$$2K = \frac{b^2 \sin C \sin A}{\sin B}, \tag{71}$$

and

$$2K = \frac{c^2 \sin A \sin B}{\sin C}. \tag{72}$$

CASE III. *Given the three sides.*

By (**67**), $2K = bc \sin A$

$$= 2bc \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \quad \text{(Art. 74)}.$$

Dividing by 2, and substituting the values of $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ from Art. 118, we have

$$K = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}. \tag{73}$$

IX. SOLUTION OF OBLIQUE TRIANGLES.

120. In the solution of plane oblique triangles we may distinguish four cases:

1. *Given a side and any two angles.*
2. *Given two sides and their included angle.*
3. *Given the three sides.*
4. *Given two sides and the angle opposite to one of them.*

Case I.

121. *Given a side and any two angles.*

The third angle may be found by Geometry, and then by aid of Art. 113 the remaining sides may be calculated.

The triangle is always possible for any values of the given elements, provided the sum of the given angles is $< 180°$.

1. Given $b = 20$, $A = 104°$, $B = 19°$. Find a, c, and C.

$$C = 180° - (A + B) = 180° - 123° = 57°.$$

By Art. 113, $\frac{a}{b} = \frac{\sin A}{\sin B}$, and $\frac{c}{b} = \frac{\sin C}{\sin B}$.

That is, $a = b \sin A \csc B$, and $c = b \sin C \csc B$.

Whence, $\log a = \log b + \log \sin A + \log \csc B$,

and $\log c = \log b + \log \sin C + \log \csc B$.

$\log b = 1.3010$	$\log b = 1.3010$
$\log \sin A = 9.9869 - 10$	$\log \sin C = 9.9236 - 10$
$\log \csc B = 0.4874$	$\log \csc B = 0.4874$
$\log a = 1.7753$	$\log c = 1.7120$
$\therefore\ a = 59.61.$	$\therefore\ c = 51.52.$

Note. To find the log cosecant of an angle, subtract the log sine from $10 - 10$. To find $\log \sin 104°$, take either $\log \cos 14°$ or $\log \sin 76°$. (See pages 7 and 10 of the explanation of the tables.)

EXAMPLES.

Solve the following triangles:

2. Given $a = 10$, $A = 38°$, $B = 77° 10'$.

3. Given $b = .8037$, $B = 52° 20'$, $C = 101° 40'$.

4. Given $c = .032$, $A = 36° 8'$, $B = 44° 27'$.

5. Given $b = 29.01$, $A = 87° 40'$, $C = 33° 15'$.

6. Given $a = 5.42$, $B = 98° 22'$, $C = 41° 1'$.

7. Given $c = .0161$, $A = 35° 15'$, $C = 123° 39'$.

8. Given $a = 400$, $A = 54° 28'$, $C = 60°$.

9. Given $b = 314.29$, $A = 67° 22'$, $B = 57° 51'$.

10. Given $c = 7.86$, $B = 32° 2' 52''$, $C = 43° 25' 26''$.

Case II.

122. *Given two sides and their included angle.*

Since one angle is known, the sum of the remaining angles may be found, and then their difference may be calculated by aid of Art. 114.

Knowing the sum and difference of the angles, the angles themselves may be obtained, and then the remaining side may be computed as in Case I.

The triangle is possible for any values of the data.

1. Given $a = 167$, $c = 82$, $B = 98°$. Find A, C, and b.

By Geometry, $$A + C = 180° - B = 82°.$$

By Art. 114, $$\frac{a+c}{a-c} = \frac{\tan\frac{1}{2}(A+C)}{\tan\frac{1}{2}(A-C)},$$

or, $$\tan\tfrac{1}{2}(A-C) = \frac{a-c}{a+c}\tan\tfrac{1}{2}(A+C).$$

Whence, $$\log\tan\tfrac{1}{2}(A-C) = \log(a-c) + \operatorname{colog}(a+c) + \log\tan\tfrac{1}{2}(A+C).$$

$$
\begin{array}{ll}
a - c = 85 & \log = 1.9294 \\
a + c = 249 & \text{colog} = 7.6038 \\
\frac{1}{2}(A + C) = 41^\circ & \log\tan = 9.9392 \\
\end{array}
$$

$$\log\tan\tfrac{1}{2}(A - C) = 9.4724$$

$$\therefore\ \tfrac{1}{2}(A - C) = 16^\circ\ 31.7'.$$

Hence, $A = \frac{1}{2}(A + C) + \frac{1}{2}(A - C) = 57^\circ\ 31.7'$,

and $C = \frac{1}{2}(A + C) - \frac{1}{2}(A - C) = 24^\circ\ 28.3'$.

To find the remaining side, we have by Art 113,

$$b = \frac{a \sin B}{\sin A} = a \sin B \csc A.$$

Whence, $\log b = \log a + \log\sin B + \log\csc A.$

$$
\begin{aligned}
\log a &= 2.2227 \\
\log\sin B &= 9.9958 \\
\log\csc A &= 0.0739 \\
\hline
\log b &= 2.2924 \\
\therefore\ b &= 196.05.
\end{aligned}
$$

EXAMPLES.

Solve the following triangles :

2. Given $a = 27$, $c = 15$, $B = 46^\circ$.

3. Given $a = 486$, $b = 347$, $C = 51^\circ\ 36'$.

4. Given $b = 2.302$, $c = 3.567$, $A = 62^\circ$.

5. Given $a = .3$, $b = .363$, $C = 124^\circ\ 56'$.

6. Given $b = 1192.1$, $c = 356.3$, $A = 26^\circ\ 16'$.

7. Given $a = 7.4$, $c = 11.439$, $B = 82^\circ\ 26'$.

8. Given $a = 53.27$, $b = 41.61$, $C = 78^\circ\ 33'$.

9. Given $b = .02668$, $c = .05092$, $A = 115^\circ\ 47'$.

10. Given $a = 51.38$, $c = 67.94$, $B = 79^\circ\ 12'\ 34''$.

Case III.

123. *Given the three sides.*

The angles might be calculated by the formulæ of Art. 117; but as these are not adapted to logarithmic computation, it is more convenient to use the formulæ of Art. 118.

Each of the three angles should be computed trigonometrically, as we then have a check on the work, since their sum should be 180°.

If all the angles are to be computed, the *tangent* formulæ are the most convenient, as only four different logarithms are required. If but one angle is required, the *cosine* formulæ will be found to involve the least work.

The triangle is possible for any values of the data, provided no side is greater than the sum of the other two.

If all the angles are required, and the tangent formulæ are used, they may be conveniently modified as follows:

By Art. 118, $\tan\frac{1}{2}A = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Denoting $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ by r, we have

$$\tan\frac{1}{2}A = \frac{r}{s-a}.$$

In like manner,

$$\tan\frac{1}{2}B = \frac{r}{s-b}, \text{ and } \tan\frac{1}{2}C = \frac{r}{s-c}.$$

1. Given $a = 2.5$, $b = 2.79$, $c = 2.33$; find A, B, and C.

In this case, $2s = a + b + c = 7.62$, and $s = 3.81$.

Whence, $s - a = 1.31$, $s - b = 1.02$, and $s - c = 1.48$.

By logarithms, we have

$$\log r = \frac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) + \text{colog}\, s].$$

Also, $\log \tan \frac{1}{2} A = \log r - \log (s - a)$,

$\log \tan \frac{1}{2} B = \log r - \log (s - b)$,

$\log \tan \frac{1}{2} C = \log r - \log (s - c)$.

$$\begin{aligned} \log (s - a) &= 0.1173 \\ \log (s - b) &= 0.0086 \\ \log (s - c) &= 0.1703 \\ \text{colog } s &= 9.4191 - 10 \\ 2)\underline{9.7153 - 10} \\ \therefore \log r &= 9.8576 - 10 \end{aligned}$$

$$\begin{aligned} \log r &= 9.8576 - 10 \\ \log (s - b) &= \underline{0.0086} \\ \log \tan \tfrac{1}{2} B &= 9.8490 - 10 \\ \tfrac{1}{2} B &= 35° \ 14.1' \\ \therefore B &= 70° \ 28.2'. \end{aligned}$$

$$\begin{aligned} \log r &= 9.8576 - 10 \\ \log (s - a) &= \underline{0.1173} \\ \log \tan \tfrac{1}{2} A &= 9.7403 - 10 \\ \tfrac{1}{2} A &= 28° \ 48.3' \\ \therefore A &= 57° \ 36.6'. \end{aligned}$$

$$\begin{aligned} \log r &= 9.8576 - 10 \\ \log (s - c) &= \underline{0.1703} \\ \log \tan \tfrac{1}{2} C &= 9.6873 - 10 \\ \tfrac{1}{2} C &= 25° \ 57.2' \\ \therefore C &= 51° \ 54.4'. \end{aligned}$$

Check, $A + B + C = 179° \ 59.2'$.

2. Given $a = 7$, $b = 11$, $c = 9.6$; find B.

By Art. 118, $\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}$.

Or, $\log \cos \frac{1}{2} B = \frac{1}{2} [\log s + \log (s - b) + \text{colog } c + \text{colog } a]$.

In this case, $2s = a + b + c = 27.6$.

Whence, $s = 13.8$, and $s - b = 2.8$.

$$\begin{aligned} \log s &= 1.1399 \\ \log (s - b) &= 0.4472 \\ \text{colog } c &= 9.0177 - 10 \\ \text{colog } a &= 9.1549 - 10 \\ 2)\underline{19.7597 - 20} \\ \log \cos \tfrac{1}{2} B &= 9.8798 - 10 \\ \tfrac{1}{2} B &= 40° \ 41.8' \\ \therefore B &= 81° \ 23.6'. \end{aligned}$$

EXAMPLES.

Solve the following triangles:

3. Given $a=2,\quad b=3,\quad c=4.$

4. Given $a=4,\quad b=7,\quad c=6.$

5. Given $a=5.6,\quad b=4.3,\quad c=4.9.$

6. Given $a=.23,\quad b=.26,\quad c=.198.$

7. Given $a=79.3,\quad b=94.2,\quad c=66.9.$

8. Given $a=321,\quad b=361,\quad c=402.$

9. Given $a=.641,\quad b=.529,\quad c=.702.$

10. Given $a=3.019,\quad b=6.731,\quad c=4.228.$

Case IV.

124. *Given two sides and the angle opposite to one of them.*

It was stated in Art. 106 that a triangle is in general completely determined when three of its elements are known, provided one of them is a side. The only exceptions occur in Case IV.

To illustrate, let us consider the following:

1. Given $a=52.1$, $b=61.2$, $A=31°\ 26'$. Required B, C, and c.

By Art. 113, $$\frac{\sin B}{\sin A}=\frac{b}{a},\quad\text{or}\quad \sin B=\frac{b\sin A}{a}.$$

Whence, $$\log\sin B=\log b+\operatorname{colog} a+\log\sin A.$$

$$\begin{aligned}\log b &= 1.7868\\ \operatorname{colog} a &= 8.2832-10\\ \log\sin A &= \underline{9.7173-10}\\ \log\sin B &= 9.7873-10\end{aligned}$$

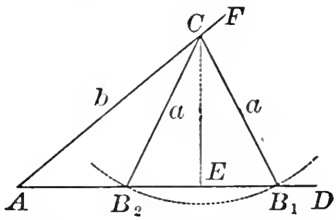
$\therefore\ B=37°\ 47.5'$, from the table.

But in determining the angle corresponding, attention must be paid to the fact that an angle and its supplement have the same sine (Art. 47).

Therefore another value of B will be $180° - 37° 47.5'$, or $142° 12.5'$; and calling these values B_1 and B_2, we have

$$B_1 = 37° 47.5', \text{ and } B_2 = 142° 12.5'.$$

Note. The reason for this ambiguity is at once apparent when we attempt to construct the triangle from the data.



We first lay off the angle DAF equal to $31° 26'$, and on AF take $AC = 61.2$. With C as a centre, and a radius equal to 52.1, describe an arc cutting AD at B_1 and B_2. Then either of the triangles AB_1C or AB_2C satisfies the given conditions.

The two values of B which were obtained are the values of the angles AB_1C and AB_2C respectively; and it is evident geometrically that these angles are supplementary.

To complete the solution, denote the angles ACB_1 and ACB_2 by C_1 and C_2, and the sides AB_1 and AB_2 by c_1 and c_2.

Then, $C_1 = 180° - (A + B_1) = 180° - 69° 13.5' = 110° 46.5'$,

and $C_2 = 180° - (A + B_2) = 180° - 173° 38.5' = 6° 21.5'$.

Again, $\dfrac{c_1}{a} = \dfrac{\sin C_1}{\sin A}$, and $\dfrac{c_2}{a} = \dfrac{\sin C_2}{\sin A}$.

Whence, $c_1 = a \sin C_1 \csc A$, and $c_2 = a \sin C_2 \csc A$.

$\log a = 1.7168$	$\log a = 1.7168$
$\log \sin C_1 = 9.9708$	$\log \sin C_2 = 9.0443$
$\log \csc A = 0.2827$	$\log \csc A = 0.2827$
$\log c_1 = 1.9703$	$\log c_2 = 1.0438$
$\therefore\ c_1 = 93.4.$	$\therefore\ c_2 = 11.06,$

125. Whenever an angle of an oblique triangle is determined from its *sine*, both the acute and obtuse values must be retained as solutions, unless one of them can be shown by other considerations to be inadmissible; and hence there may sometimes be two solutions, sometimes only one, and sometimes none, in an example under Case IV.

I. Let the data be a, b, and A, and suppose $b < a$.

Since, by Geometry, B must be $< A$, only the *acute* value of B can be taken; in this case there is but *one* solution.

II. Let the data be a, b, and A, and suppose $b > a$.

Since B must be $> A$, the triangle is impossible unless A is acute.

Again, since $\frac{\sin B}{\sin A} = \frac{b}{a}$, and b is $> a$, $\sin B$ is $> \sin A$.

Hence both the acute and obtuse values of B are $> A$, and there are *two* solutions, except in the following cases:

If the data are such as to make $\log \sin B = 0$, then $\sin B = 1$ (Art. 87) and $B = 90°$, and the triangle is a right triangle; if $\log \sin B$ is *positive*, then $\sin B$ is > 1, and the triangle is impossible.

126. The results of Art. 125 may be stated as follows:

If, of the given sides, that adjacent to the given angle is the *less*, there is but *one* solution, corresponding to the *acute* value of the opposite angle.

If the side adjacent to the given angle is the *greater*, there are *two* solutions unless the log sine of the opposite angle is 0 or positive; in which cases there are *one* solution (a *right* triangle), and *no* solution, respectively.

127. We will illustrate the above points by examples:

2. Given $a = 7.42$, $b = 3.39$, $A = 105° \, 13'$; find B.

Since b is $< a$, there is but *one* solution, corresponding to the *acute* value of B.

We have, $\sin B = \dfrac{b \sin A}{a}.$

$$\begin{aligned}
\log b &= 0.5302\\
\text{colog } a &= 9.1296\\
\log \sin A &= 9.9845\\
\hline
\log \sin B &= 9.6443\\
\therefore B &= 26^\circ\ 9.6'.
\end{aligned}$$

3. Given $b = 3$, $c = 2$, $C = 100^\circ$; find B.

Since b is $> c$, and C is obtuse, the triangle is impossible.

4. Given $a = 22.764$, $c = 50$, $A = 27^\circ\ 4.8'$; find C.

We have, $\sin C = \dfrac{c \sin A}{a}.$

$$\begin{aligned}
\log c &= 1.6990\\
\text{colog } a &= 8.6428\\
\log \sin A &= 9.6582\\
\hline
\log \sin C &= 0.0000\\
\therefore \sin C &= 1, \text{ and } C = 90^\circ.
\end{aligned}$$

Here there is but one solution; a *right* triangle.

5. Given $a = .83$, $b = .715$, $B = 61^\circ\ 47'$; find A.

We have, $\sin A = \dfrac{a \sin B}{b}.$

$$\begin{aligned}
\log a &= 9.9191\\
\text{colog } b &= 0.1457\\
\log \sin B &= 9.9451\\
\hline
\log \sin A &= 0.0099
\end{aligned}$$

Since log sin A is positive, the triangle is impossible.

EXAMPLES.

Solve the following triangles:

6. Given $a = 5.08$, $b = 3.59$, $A = 63^\circ\ 50'$.

7. Given $b = 74.8$, $c = 62.2$, $C = 27^\circ\ 18'$.

8.	Given $b = .2337$,	$c = .1982$,	$B = 109°$.
9.	Given $a = 1.07$,	$c = 1.71$,	$C = 31° 53'$.
10.	Given $a = .1864$,	$b = .17$,	$B = 63° 40'$.
11.	Given $a = 50$,	$c = 66$,	$A = 123° 11'$.
12.	Given $b = 50.3$,	$c = 66.8$,	$C = 32° 49'$.
13.	Given $a = 8.656$,	$c = 10$,	$A = 59° 57'$.
14.	Given $b = 5.161$,	$c = 6.84$,	$B = 44° 3'$.
15.	Given $a = 214.56$,	$b = 284.79$,	$B = 104° 20'$.
16.	Given $b = 3069$,	$c = 1223$,	$C = 55° 52'$.
17.	Given $a = .7097$,	$c = .5112$,	$A = 35° 11'$.
18.	Given $a = 106.85$,	$b = 166.21$,	$A = 40° 0' 21''$.
19.	Given $a = .3216$,	$c = .2708$,	$C = 52° 24'$.
20.	Given $b = 811.3$,	$c = 606.4$,	$B = 126° 5' 20''$.

AREA OF AN OBLIQUE TRIANGLE.

128. 1. Given $a = 18.063$, $A = 96° 30'$, $B = 35°$; find K.

By Art. 119,
$$2K = \frac{a^2 \sin B \sin C}{\sin A}$$
$$= a^2 \sin B \sin C \csc A.$$

Whence,
$$\log(2K) = 2 \log a + \log \sin B + \log \sin C + \log \csc A.$$

From the data,
$$C = 180° - (A + B) = 48° 30'.$$

$$\begin{aligned} \log a = 1.2568;\ \text{multiply by } 2 &= 2.5136 \\ \log \sin B &= 9.7586 \\ \log \sin C &= 9.8745 \\ \log \csc A &= \underline{0.0028} \\ \log(2K) &= 2.1495 \end{aligned}$$

$$\therefore\ 2K = 141.1, \text{ and } K = 70.55.$$

EXAMPLES.

Find the areas of the following triangles:

2. Given $a = 38$,	$c = 61.2$,	$B = 67° 56'$.
3. Given $a = 5$,	$b = 7$,	$c = 6$.
4. Given $b = 2.07$,	$A = 70°$,	$B = 36° 23'$.
5. Given $b = 116.1$,	$c = 100$,	$A = 118° 16'$.
6. Given $a = 79$,	$b = 94$,	$c = 67$.
7. Given $a = 3.123$,	$A = 53° 11'$,	$B = 13° 57'$.
8. Given $b = .439$,	$A = 76° 38'$,	$C = 40° 35'$.
9. Given $a = 23.1$,	$b = 19.7$,	$c = 25.2$.
10. Given $a = .3228$,	$c = .9082$,	$B = 60° 16'$.
11. Given $c = 80.25$,	$B = 100° 5'$,	$C = 31° 44'$.
12. Given $a = .010168$,	$b = .018225$,	$C = 11° 18' 26''$.
13. Given $a = 5.82$,	$b = 6$,	$c = 4.26$.

MISCELLANEOUS EXAMPLES.

129. **1.** From a point in the same horizontal plane with the base of a tower, the angle of elevation of its top is 52° 39′, and from a point 100 feet further away it is 35° 16′. Required the height of the tower, and its distance from each of the points of observation.

2. In a field $ABCD$, the sides AB, BC, CD, and DA are 155, 236, 252, and 105 rods, respectively, and the length of a line from A to C is 311 rods. Find the area of the field.

3. From the top of a bluff, the angles of depression of two posts in the plain below, in line with the observer and 1000 feet apart, are found to be 27° 40′ and 9° 33′, respectively. What is the height of the bluff above the plain?

4. Two yachts start at the same time from the same point, and sail, one due north at the rate of 10.44 miles an

hour, and the other due northeast at the rate of 7.71 miles an hour. How far apart are they at the end of 40 minutes?

5. A ship is sailing due southwest at the rate of 8 miles an hour. At 10.30 A.M., a lighthouse is observed to bear 30° west of north, and at 12.15 P.M., it is observed to bear 15° east of north. Find the distance of the lighthouse from each position of the ship.

6. Wishing to find the distance of an inaccessible object A from a position B, I measure a line BC, 208.3 feet in length. The angles ABC and ACB are measured, and found to be 126° 35′ and 31° 48′, respectively. Required the distance AB.

7. A flagpole 40 feet in height stands on the top of a tower. From a position near the base of the tower, the angles of elevation of the top and bottom of the pole are 38° 53′ and 20° 18′, respectively. Required the distance and height of the tower.

8. A surveyor observes that his position A is exactly in line with two inaccessible objects B and C. He measures a line AD, 500 feet in length, making the angle $BAD = 60°$, and at D observes the angles ADB and BDC to be 40° and 60°, respectively. Required the distance BC.

9. To find the distance between two buoys A and B, I measure a base line CD on the shore, 150 feet in length. At the point C the angles ACD and BCD are measured and found to be 95° and 70°, respectively; and at D the angles BDC and ADC are found to be 83° and 30°. What is the distance between the buoys?

10. The sides of a field $ABCD$ are $AB = 37$, $BC = 63$, and $DA = 20$, and the diagonals AC and BD are 75 and 42, respectively. Required the area of the field.

PART II.

SPHERICAL TRIGONOMETRY.

X. GEOMETRICAL DEFINITIONS AND PRINCIPLES.

130. If a triedral angle is formed with its vertex at the centre of a sphere, it intercepts on the surface a *spherical triangle*.

131. The triangle is bounded by three arcs of great circles called its sides, which measure the face angles of the triedral angle.

The *angles* of the spherical triangle are the diedral angles of the triedral angle; and by Geometry, each is measured by the angle between two straight lines drawn, one in each face, and perpendicular to the edge at the same point.

132. The sides of a spherical triangle, being arcs, are usually expressed in degrees.

If the length of a side in terms of some *linear* unit is desired, it may be obtained by finding the ratio of its arc to 360°, and multiplying the result by the length of the circumference of a great circle.

133. *Spherical Trigonometry* treats of the trigonometrical relations between the elements of a spherical triangle; or what is the same thing, between the face and diedral angles of the triedral angle which intercepts it.

134. The face and diedral angles are not altered in magnitude by varying the radius of the sphere, and hence the relations between the sides and angles of a spherical triangle are independent of the length of the radius.

135. We shall limit ourselves in this work to such triangles as are considered in Geometry, where each angle is less than two right angles, and each side less than the semicircumference of a great circle; that is, where each element is less than 180°.

136. The proofs of the following properties of spherical triangles may be found in any treatise on Solid Geometry:

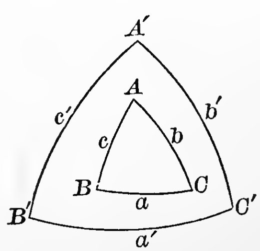
(*a*) Either side of a spherical triangle is less than the sum of the other two sides.

(*b*) If two sides of a spherical triangle are unequal, the angles opposite them are unequal, and the greater angle lies opposite the greater side; and conversely.

(*c*) The sum of the sides of a spherical triangle is less than 360°.

(*d*) The sum of the angles of a spherical triangle is greater than 180°, and less than 540°.

(*e*) If $A'B'C'$ is the polar triangle of ABC, *i.e.*, if A, B, and C are the poles of the arcs a', b', and c', respectively, then conversely, ABC is the polar triangle of $A'B'C'$.



(*f*) In two polar triangles, each angle of one is measured by the supplement of the side lying opposite to it in the other.

That is,

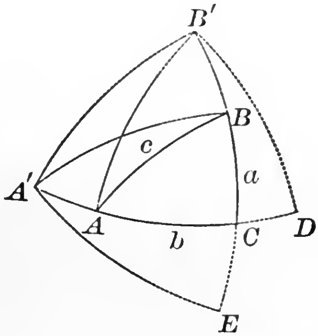
$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c',$$
$$A' = 180° - a, \quad B' = 180° - b, \quad C' = 180° - c.$$

137. A spherical triangle is called *tri-rectangular* when it has three right angles; each of its sides is a quadrant, and each vertex is the pole of the opposite side.

138. I. Let C be the right angle of the spherical right triangle ABC, and suppose $a < 90°$ and $b < 90°$.

Complete the tri-rectangular triangle $A'B'C$.

Also, since B' is the pole of AC, and A' of BC, construct the tri-rectangular triangles $AB'D$ and $A'BE$.

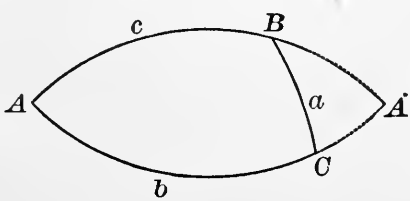


Then since A and B lie on the same side of $B'D$, AB or c is $< 90°$.

Since BC is $< B'C$, the angle A is $< B'AD$, or $< 90°$.

Since AC is $< A'C$, the angle B is $< A'BE$, or $< 90°$.

II. Suppose $a < 90°$ and $b > 90°$.



Complete the lune $ABA'C$.

Then in the right triangle $A'BC$, $A'C = 180° - b$.

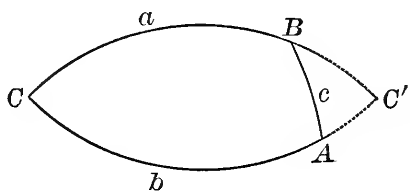
That is, the sides a and $A'C$ of the triangle $A'BC$ are each $< 90°$; and by I., $A'B$ and the angles A' and $A'BC$ are each $< 90°$.

But, $c = 180° - A'B$, $A = A'$, and $B = 180° - A'BC$.

Whence, c is $> 90°$, $A < 90°$, and $B > 90°$.

In like manner, if a is $> 90°$ and $b < 90°$, then c is $> 90°$, $A > 90°$, and $B < 90°$.

III. Suppose $a > 90°$ and $b > 90°$.



Complete the lune $ACBC'$.

Then in the right triangle ABC', $AC' = 180° - b$, and $BC' = 180° - a$.

That is, the sides AC' and BC' of the triangle ABC' are each $< 90°$; and by I., AB and the angles BAC' and ABC' are each $< 90°$.

But, $A = 180° - BAC'$, and $B = 180° - ABC'$.

Whence, c is $< 90°$, $A > 90°$, and $B > 90°$.

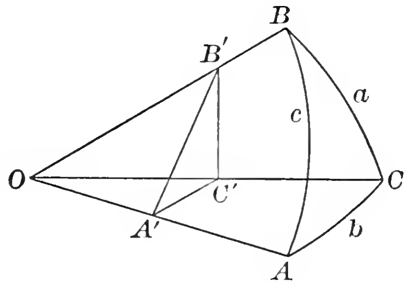
Hence, in any spherical right triangle:

1. *If the sides including the right angle are in the same quadrant, the hypotenuse is* $< 90°$; *if they are in different quadrants, the hypotenuse is* $> 90°$.

2. *An angle is in the same quadrant as its opposite side.*

XI. SPHERICAL RIGHT TRIANGLES.

139. Let C be the right angle of the spherical right triangle ABC, and O the centre of the sphere.



Join OA, OB, and OC.

At any point A' of OA draw $A'B'$ and $A'C'$ perpendicular to OA, and join $B'C'$.

Then by Art. 131, the sides a, b, and c measure the angles BOC, COA, and AOB, respectively, and the angle $B'A'C'$ is equal to the angle A of the spherical triangle.

Since OA is perpendicular to $A'B'$ and $A'C'$, it is perpendicular to the plane $A'B'C'$.

Whence, since each of the planes $A'B'C'$ and OBC is perpendicular to the plane OAC, their intersection $B'C'$ is perpendicular to OAC.

Therefore $B'C'$ is perpendicular to $A'C'$ and OC'.

In the right triangle $OA'B'$, we have

$$\cos c = \cos A'OB' = \frac{OA'}{OB'} = \frac{OC'}{OB'} \times \frac{OA'}{OC'}.$$

But in the right triangles $OB'C'$ and $OC'A'$,

$$\frac{OC'}{OB'} = \cos a, \text{ and } \frac{OA'}{OC'} = \cos b.$$

Whence, $\cos c = \cos a \cos b$. **(74)**

Again, $$\sin A = \sin B'A'C' = \frac{B'C'}{A'B'} = \frac{\dfrac{B'C'}{OB'}}{\dfrac{A'B'}{OB'}} = \frac{\sin a}{\sin c}, \qquad (75)$$

and $$\cos A = \cos B'A'C' = \frac{A'C'}{A'B'} = \frac{\dfrac{A'C'}{OA'}}{\dfrac{A'B'}{OA'}} = \frac{\tan b}{\tan c}. \qquad (76)$$

In like manner we have,

$$\sin B = \frac{\sin b}{\sin c}, \qquad (77)$$

and $$\cos B = \frac{\tan a}{\tan c}. \qquad (78)$$

140. From (**75**) and (**76**), we obtain

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin a}{\sin c} \times \frac{\tan c}{\tan b} = \frac{\sin a}{\cos c \tan b}.$$

Whence by (**74**),

$$\tan A = \frac{\sin a}{\cos a \cos b \tan b} = \frac{\tan a}{\sin b}. \qquad (79)$$

Similarly,

$$\tan B = \frac{\tan b}{\sin a}. \qquad (80)$$

141. Since $\sin a = \cos a \tan a$ (Art. 20), (**75**) may be written

$$\sin A = \frac{\cos a \tan a}{\cos c \tan c} = \frac{\dfrac{\tan a}{\tan c}}{\dfrac{\cos c}{\cos a}}.$$

Whence by (**74**) and (**78**),

$$\sin A = \frac{\cos B}{\cos b}. \qquad (81)$$

Similarly,

$$\sin B = \frac{\cos A}{\cos a}. \qquad (82)$$

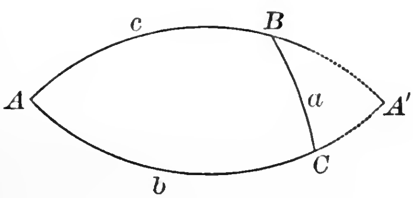
142. From (**74**), (**81**), and (**82**), we have

$$\begin{aligned} \cos c &= \cos a \cos b \\ &= \frac{\cos A}{\sin B} \times \frac{\cos B}{\sin A} \\ &= \cot A \cot B. \qquad (83) \end{aligned}$$

143. The proofs of Art. 139 cannot be regarded as general, for in the construction of the figure we have assumed a and b, and therefore c and A (Art. 138), to be less than 90°.

To prove formulæ (**74**) to (**78**) universally, it is necessary to consider two additional cases:

Case I. *When one of the sides a and b is $< 90°$, and the other $> 90°$.*



In the right triangle ABC, let a be $< 90°$ and $b > 90°$. Complete the lune $ABA'C$; then in the triangle $A'BC$,

$$A'B = 180° - c, \qquad A' = A,$$

$$A'C = 180° - b, \qquad A'BC = 180° - B.$$

But by Art. 138, c is $> 90°$, $A < 90°$, and $B > 90°$. Hence each element, except the right angle, of the right triangle $A'BC$ is $< 90°$, and we have by Art. 139,

$$\cos A'B = \cos a \cos A'C,$$

$$\sin A' = \frac{\sin a}{\sin A'B}, \qquad \sin A'BC = \frac{\sin A'C}{\sin A'B},$$

$$\cos A' = \frac{\tan A'C}{\tan A'B}, \qquad \cos A'BC = \frac{\tan a}{\tan A'B}.$$

Putting for $A'B$, $A'C$, A', and $A'BC$ their values, we have

$$\cos(180^\circ - c) = \cos a \cos(180^\circ - b),$$

$$\sin A = \frac{\sin a}{\sin(180^\circ - c)}, \quad \sin(180^\circ - B) = \frac{\sin(180^\circ - b)}{\sin(180^\circ - c)},$$

$$\cos A = \frac{\tan(180^\circ - b)}{\tan(180^\circ - c)}, \quad \cos(180^\circ - B) = \frac{\tan a}{\tan(180^\circ - c)}.$$

Whence by Art. 47,

$$-\cos c = \cos a(-\cos b),$$

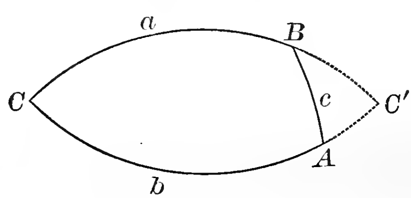
$$\sin A = \frac{\sin a}{\sin c}, \qquad \sin B = \frac{\sin b}{\sin c},$$

$$\cos A = \frac{-\tan b}{-\tan c}, \qquad -\cos B = \frac{\tan a}{-\tan c};$$

and we obtain the formulæ (**74**) to (**78**) as before.

In like manner, the formulæ may be proved in the case where a is $> 90^\circ$ and $b < 90^\circ$.

CASE II. *When both a and b are $> 90^\circ$.*



In the right triangle ABC, let a and b be $> 90^\circ$.

Complete the lune $ACBC'$.

By Art. 138, c is $< 90^\circ$, $A > 90^\circ$, and $B > 90^\circ$.

Hence each element, except the right angle, of the right triangle ABC' is $< 90^\circ$, and we have by Art. 139,

$$\cos c = \cos AC' \cos BC',$$

$$\sin BAC' = \frac{\sin BC'}{\sin c}, \qquad \sin ABC' = \frac{\sin AC'}{\sin c},$$

$$\cos BAC' = \frac{\tan AC'}{\tan c}, \qquad \cos ABC' = \frac{\tan BC'}{\tan c}.$$

Putting for AC', BC', BAC', and ABC' their values, we have

$$\cos c = \cos(180^\circ - a)\cos(180^\circ - b),$$

$$\sin(180^\circ - A) = \frac{\sin(180^\circ - a)}{\sin c},\quad \sin(180^\circ - B) = \frac{\sin(180^\circ - b)}{\sin c},$$

$$\cos(180^\circ - A) = \frac{\tan(180^\circ - b)}{\tan c},\quad \cos(180^\circ - B) = \frac{\tan(180^\circ - a)}{\tan c}.$$

Whence by Art. 47,

$$\cos c = (-\cos a)(-\cos b),$$

$$\sin A = \frac{\sin a}{\sin c},\qquad \sin B = \frac{\sin b}{\sin c},$$

$$-\cos A = \frac{-\tan b}{\tan c},\qquad -\cos B = \frac{-\tan a}{\tan c};$$

and we obtain the formulæ (**74**) to (**78**) as before.

144. The formulæ of Arts. 139 to 142 are collected below for the convenience of the student:

$$\cos c = \cos a \cos b.$$

$$\sin A = \frac{\sin a}{\sin c}.\qquad \sin B = \frac{\sin b}{\sin c}.$$

$$\cos A = \frac{\tan b}{\tan c}.\qquad \cos B = \frac{\tan a}{\tan c}.$$

$$\tan A = \frac{\tan a}{\sin b}.\qquad \tan B = \frac{\tan b}{\sin a}.$$

$$\sin A = \frac{\cos B}{\cos b}.\qquad \sin B = \frac{\cos A}{\cos a}.$$

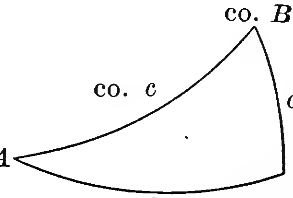
$$\cos c = \cot A \cot B.$$

By comparing the formulæ for the sines, cosines, and tangents of A and B with the corresponding forms for plane triangles as given in Arts. 10 and 14, no difficulty will be found in retaining them in the memory.

NAPIER'S RULES OF CIRCULAR PARTS.

145. These are two artificial rules which include all the formulæ of the preceding article.

In any spherical right triangle, the elements a and b, and the complements of the elements A, B, and c (written in abbreviated form, co. A, co. B, and co. c), are called the *circular parts.*



If we suppose them arranged in the order in which the letters occur in the triangle, any one of the five may be taken and called the *middle part;* the two immediately adjacent are called the *adjacent parts*, and the remaining two the *opposite parts.*

Then Napier's rules are:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

146. Napier's rules may be proved by taking each of the circular parts in succession as the middle part, and showing that the results agree with the formulæ of Art. 144.

1. If a is the middle part, b and co. B are the adjacent parts, and co. c and co. A the opposite parts. Then the rules give

$$\sin a = \tan b \tan(\text{co. } B),$$

and $$\sin a = \cos(\text{co. } c) \cos(\text{co. } A);$$

or, by Art. 14, $\sin a = \tan b \cot B$, and $\sin a = \sin c \sin A$;

which agree with (**80**) and (**75**).

2. If b is the middle part, a and co. A are the adjacent parts, and co. c and co. B the opposite parts. Then,

$$\sin b = \tan a \tan (\text{co. } A)$$
$$= \tan a \cot A,$$

and
$$\sin b = \cos (\text{co. } c) \cos (\text{co. } B)$$
$$= \sin c \sin B;$$

which agree with **(79)** and **(77)**.

3. If co. c is the middle part, co. A and co. B are the adjacent parts, and a and b the opposite parts. Then,

$$\sin (\text{co. } c) = \tan (\text{co. } A) \tan (\text{co. } B),$$

or
$$\cos c = \cot A \cot B;$$

and
$$\sin (\text{co. } c) = \cos a \cos b,$$

or
$$\cos c = \cos a \cos b;$$

which agree with **(83)** and **(74)**.

4. If co. A is the middle part, b and co. c are the adjacent parts, and a and co. B the opposite parts. Then,

$$\sin (\text{co. } A) = \tan b \tan (\text{co. } c), \quad \text{or} \quad \cos A = \tan b \cot c,$$

and
$$\sin (\text{co. } A) = \cos a \cos (\text{co. } B), \quad \text{or} \quad \cos A = \cos a \sin B;$$

which agree with **(76)** and **(82)**.

5. If co. B is the middle part, a and co. c are the adjacent parts, and b and co. A the opposite parts. Then,

$$\sin (\text{co. } B) = \tan a \tan (\text{co. } c), \quad \text{or} \quad \cos B = \tan a \cot c,$$

and
$$\sin (\text{co. } B) = \cos b \cos (\text{co. } A), \quad \text{or} \quad \cos B = \cos b \sin A;$$

which agree with **(78)** and **(81)**.

147. Writers on Trigonometry differ as to the practical value of Napier's rules; but in the opinion of the highest authorities, it seems to be regarded as preferable to attempt to remember the formulæ by comparing them with the analogous forms for plane triangles, as stated in Art. 144.

SOLUTION OF SPHERICAL RIGHT TRIANGLES.

148. To solve a spherical right triangle, two elements must be given in addition to the right angle.

There may be six cases:

1. *Given the hypotenuse and an adjacent angle.*
2. *Given an angle and its opposite side.*
3. *Given an angle and its adjacent side.*
4. *Given the hypotenuse and another side.*
5. *Given the two sides a and b.*
6. *Given the two angles A and B.*

149. Either of the above may be solved by aid of Art. 144.

The formula for computing either of the remaining elements when any two are given may be found by the following rule:

Take that formula which involves the given parts and the required part.

If *all* the remaining elements are required, the following rule may be found convenient in selecting the formulæ:

Take the three formulæ which involve the given parts.

150. It is convenient in the solution to have a check on the logarithmic work, which may be done in every case without the necessity of looking out any new logarithms.

Examples of this will be found in Art. 153.

The check formula for any particular case may be selected from the set in Art. 144 by the following rule:

Take that formula which involves the three required parts.

Note. If Napier's rules are used, the following rule will indicate which of the circular parts corresponding to the given elements and any required element is to be regarded as the middle part:

If these three circular parts are adjacent, take the middle one as the middle part, and the others are then adjacent parts.

If they are not adjacent, take the part which is not adjacent to either of the others as the middle part, and the others are then the opposite parts.

For the check formula, proceed as above with the circular parts corresponding to the three required elements.

Thus if c and A are the given elements,

1. To find a, consider the circular parts a, co. c, and co. A; of these, a is the middle part, and co. c and co. A are opposite parts. Then, by Napier's rules,

$$\sin a = \cos(\text{co.}\,c)\cos(\text{co.}\,A) = \sin c \sin A.$$

2. To find b, the circular parts are b, co. c, and co. A; in this case co. A is the middle part, and b and co. c are adjacent parts. Then,

$$\sin(\text{co.}\,A) = \tan b \tan(\text{co.}\,c), \text{ or } \cos A = \tan b \cot c.$$

3. To find B, the circular parts are co. B, co. c, and co. A; co. c is the middle part, and co. A and co. B are adjacent parts. Then,

$$\sin(\text{co.}\,c) = \tan(\text{co.}\,A)\tan(\text{co.}\,B), \text{ or } \cos c = \cot A \cot B.$$

4. For the check formula, the circular parts are a, b, and co. B; a is the middle part, and b and co. B are adjacent parts. Then,

$$\sin a = \tan b \tan(\text{co.}\,B) = \tan b \cot B.$$

151. In solving spherical triangles, careful attention must be paid to the *algebraic signs* of the functions; the cosines, tangents, and cotangents of angles greater than 90° being taken *negative*.

It is convenient to place the sign of each function just above or below it, as illustrated in the examples of Art. 153; the sign of the function in the first member being then determined in accordance with the principle that like signs produce +, and unlike signs produce −.

Note. In the examples after the first of Art. 153, the signs are omitted in every case where both factors of the second member are +.

152. In finding the angles corresponding, if the function is a cosine, tangent, or cotangent, its sign determines whether the angle is less or greater than 90°; that is, if it is +, the angle is $< 90°$; and if it is −, the angle is $> 90°$, and the *supplement* of the acute angle obtained from the tables must be taken (Art. 47).

If the function is a sine, since the sine of an angle is equal to the sine of its supplement, both the acute angle obtained from the tables and its supplement must be retained as solutions, unless the ambiguity can be removed by the principles of Art. 138.

EXAMPLES.

153. 1. Given $B = 33° \, 50'$, $a = 108°$; find A, b, and c.

By the rule of Art. 149, the formulæ from Art. 144 are,

$$\sin B = \frac{\cos A}{\cos a}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \cos B = \frac{\tan a}{\tan c}.$$

That is,

$$\overset{-}{\cos A} = \overset{-}{\cos a}\, \overset{+}{\sin B}, \quad \overset{+}{\tan b} = \overset{+}{\sin a}\, \overset{+}{\tan B}, \quad \overset{-}{\tan c} = \frac{\overset{-}{\tan a}}{\underset{+}{\cos B}}.$$

Hence,
$$\begin{aligned} \log \cos A &= \log \cos a + \log \sin B, \\ \log \tan b &= \log \sin a + \log \tan B, \\ \log \tan c &= \log \tan a - \log \cos B. \end{aligned}$$

Since $\cos A$ and $\tan c$ are negative, the *supplements* of the angles obtained from the tables must be taken (Art. 152).

Note. When the supplement of the angle obtained from the tables is to be taken, it is convenient to write 180° minus the element in the first member, as shown below in the cases of A and c.

By the rule of Art. 150, the check formula for this case is

$$\cos A = \frac{\tan b}{\tan c}, \quad \text{or} \quad \log \cos A = \log \tan b - \log \tan c.$$

The values of $\log \tan b$ and $\log \tan c$ may be taken from the first part of the work, and their difference should be equal to the result previously found for $\log \cos A$.

$$\begin{aligned} \log \cos a &= 9.4900 - 10 \\ \log \sin B &= 9.7457 - 10 \\ \hline \log \cos A &= 9.2357 - 10 \\ 180° - A &= 80° \, 5.5' \\ \therefore \; A &= 99° \, 54.5'. \end{aligned}
\qquad
\begin{aligned} \log \tan a &= 0.4882 \\ \log \cos B &= 9.9194 - 10 \\ \hline \log \tan c &= 0.5688 \\ 180° - c &= 74° \, 53.8' \\ \therefore \; c &= 105° \, 6.2'. \end{aligned}$$

$$\begin{aligned} \log \sin a &= 9.9782 - 10 \\ \log \tan B &= 9.8263 - 10 \\ \hline \log \tan b &= 9.8045 - 10 \\ \therefore \; b &= 32° \, 31.1'. \end{aligned}$$

Check.

$$\begin{aligned} \log \tan b &= 9.8045 - 10 \\ \log \tan c &= 0.5688 \\ \hline \log \cos A &= 9.2357 - 10 \end{aligned}$$

2. Given $c = 70° \, 30'$, $A = 100°$; find a, b, and B.

By Art. 149, the three formulæ are,

$$\sin A = \frac{\sin a}{\sin c}, \quad \cos A = \frac{\tan b}{\tan c}, \quad \cos c = \cot A \cot B.$$

That is,

$$\sin a = \sin c \sin A, \quad \overset{-}{\tan b} = \overset{+}{\tan c}\, \overset{-}{\cos A}, \quad \overset{-}{\cot B} = \overset{+}{\cos c}\, \overset{-}{\tan A}.$$

The side a is determined from its sine; but the ambiguity is removed by the principles of Art. 138; for a and A must be in the same quadrant. Therefore a is $> 90°$, and the supplement of the angle obtained from the table must be taken.

By Art. 150, the check formula is

$$\tan B = \frac{\tan b}{\sin a}, \quad \text{or} \quad \sin a = \tan b \cot B.$$

Note 1. The check formula should always be expressed in terms of the functions used in determining the required parts; thus, in the case above, the check formula is transformed so as to involve $\cot B$ instead of $\tan B$.

$$\begin{array}{rl}
\log \sin c &= 9.9743 \\
\log \sin A &= \underline{9.9934} \\
\log \sin a &= 9.9677 \\
180° - a &= 68° \, 10' \\
\therefore \; a &= 111° \, 50'.
\end{array}
\qquad
\begin{array}{rl}
\log \cos c &= 9.5235 \\
\log \tan A &= \underline{0.7537} \\
\log \cot B &= 0.2772 \\
180° - B &= 27° \, 50.6' \\
\therefore \; B &= 152° \, 9.4'.
\end{array}$$

$$\begin{array}{rl}
\log \tan c &= 0.4509 \\
\log \cos A &= \underline{9.2397} \\
\log \tan b &= 9.6906 \\
180° - b &= 26° \, 7.5' \\
\therefore \; b &= 153° \, 52.5'.
\end{array}
\qquad
\begin{array}{rl}
& \textit{Check.} \\
\log \tan b &= 9.6906 \\
\log \cot B &= \underline{0.2772} \\
\log \sin a &= 9.9678
\end{array}$$

Note 2. We observe here a difference of .0001 in the two values of $\log \sin a$. This does not necessarily indicate an error in the work, for such a small difference might easily be due to the fact that the logarithms are only *approximately* correct to the fourth decimal place.

3. Given $a = 132° 6'$, $b = 77° 51'$; find A, B, and c.

In this case the formulæ are,

$$\overset{-}{\tan} A = \frac{\overset{-}{\tan a}}{\underset{+}{\sin b}}, \quad \tan B = \frac{\tan b}{\sin a}, \quad \overset{-}{\cos c} = \overset{-}{\cos a}\,\overset{+}{\cos b}.$$

The check formula is

$$\cos c = \cot A \cot B, \quad \text{or} \quad \cos c \tan A \tan B = 1.$$

That is, $\log \cos c + \log \tan A + \log \tan B = \log 1 = 0$

$$\begin{aligned} \log \tan a &= 0.0440 \\ \log \sin b &= \underline{9.9901} \\ \log \tan A &= 0.0539 \\ 180° - A &= 48° 32.8' \\ \therefore\ A &= 131° 27.2'. \end{aligned} \qquad \begin{aligned} \log \cos a &= 9.8263 \\ \log \cos b &= \underline{9.3232} \\ \log \cos c &= 9.1495 \\ 180° - c &= 81° 53.4' \\ \therefore\ c &= 98° 6.6'. \end{aligned}$$

$$\begin{aligned} \log \tan b &= 0.6670 \\ \log \sin a &= \underline{9.8704} \\ \log \tan B &= 0.7966 \\ \therefore\ B &= 80° 55.4'. \end{aligned}$$

Check.

$$\begin{aligned} \log \cos c &= 9.1495 \\ \log \tan A &= 0.0539 \\ \log \tan B &= \underline{0.7966} \\ \log 1 &= 0.0000 \end{aligned}$$

4. Given $A = 105° 59'$, $a = 128° 33'$; find b, B, and c.

The formulæ are,

$$\overset{+}{\sin} b = \frac{\overset{-}{\tan a}}{\underset{-}{\tan A}}, \quad \overset{+}{\sin} B = \frac{\overset{-}{\cos A}}{\underset{-}{\cos a}}, \quad \sin c = \frac{\sin a}{\sin A}.$$

The check formula is

$$\sin B = \frac{\sin b}{\sin c}.$$

In this example, each of the required parts is determined from its sine; and as the ambiguity cannot be removed by Art. 138, both the acute angle obtained from the tables and its supplement must be retained in each case.

$$\begin{aligned} \log\tan a &= 0.0986 \\ \log\tan A &= 0.5430 \\ \hline \log\sin b &= 9.5556 \\ \therefore\ b &= 21^\circ\ 3.9', \\ &\text{or } 158^\circ\ 56.1'. \end{aligned}$$

$$\begin{aligned} \log\sin a &= 9.8932 \\ \log\sin A &= 9.9828 \\ \hline \log\sin c &= 9.9104 \\ \therefore\ c &= 54^\circ\ 26.7', \\ &\text{or } 125^\circ\ 33.3'. \end{aligned}$$

$$\begin{aligned} \log\cos A &= 9.4399 \\ \log\cos a &= 9.7946 \\ \hline \log\sin B &= 9.6453 \\ \therefore\ B &= 26^\circ\ 13.5', \\ &\text{or } 153^\circ\ 46.5'. \end{aligned}$$

Check.

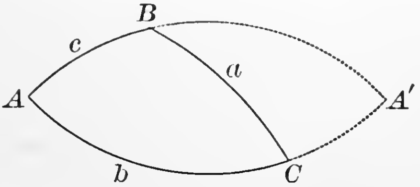
$$\begin{aligned} \log\sin b &= 9.5556 \\ \log\sin c &= 9.9104 \\ \hline \log\sin B &= 9.6452 \end{aligned}$$

It does not follow, however, that these values can be combined promiscuously; for by Art. 138, since a is $> 90^\circ$, with the value of b less than 90° must be taken the value of c greater than 90°, and the value of B less than 90°; while with the value of b greater than 90° must be taken the value of c less than 90°, and the value of B greater than 90°.

Thus the only solutions of the example are:

1. $b = 21^\circ\ 3.9'$, $c = 125^\circ\ 33.3'$, $B = 26^\circ\ 13.5'$.
2. $b = 158^\circ\ 56.1'$, $c = 54^\circ\ 26.7'$, $B = 153^\circ\ 46.5'$.

Note. The figure shows geometrically why there are two solutions in this case.



For if AB and AC are produced to A', forming the lune $ABA'C$, the triangle $A'BC$ has the side a and the angle A' equal, respectively, to the side a and the angle A of the triangle ABC, and both triangles are right-angled at C.

It is evident that the sides $A'B$ and $A'C$ and the angle $A'BC$ are the supplements of the sides c and b and the angle ABC, respectively.

Solve the following spherical right triangles:

5. Given $a = 159^\circ$, $c = 137^\circ\, 20'$.

6. Given $A = 50^\circ\, 20'$, $B = 122^\circ\, 40'$.

7. Given $a = 160^\circ$, $b = 38^\circ\, 30'$.

8. Given $B = 80^\circ$, $b = 67^\circ\, 40'$.

9. Given $B = 112^\circ$, $c = 81^\circ\, 50'$.

10. Given $a = 61^\circ$, $B = 123^\circ\, 40$.

11. Given $a = 61^\circ\, 40'$, $b = 144^\circ\, 10'$.

12. Given $A = 99^\circ\, 50'$, $a = 112^\circ$.

13. Given $b = 15^\circ$, $c = 152^\circ\, 20'$.

14. Given $A = 62^\circ\, 59'$, $B = 37^\circ\; 4'$.

15. Given $A = 73^\circ\; 7'$, $c = 114^\circ\, 32'$.

16. Given $B = 144^\circ\, 54'$, $b = 146^\circ\, 32'$.

17. Given $B = 68^\circ\, 18'$, $c = 47^\circ\, 34'$.

18. Given $A = 161^\circ\, 52'$, $b = 131^\circ\; 8'$.

19. Given $a = 113^\circ\, 25'$, $b = 110^\circ\, 47'$.

20. Given $a = 137^\circ\; 9'$, $B = 74^\circ\, 51'$.

21. Given $A = 144^\circ\, 54'$, $B = 101^\circ\, 14'$.

22. Given $a = 69^\circ\, 18'$, $c = 84^\circ\, 27'$.

SOLUTION OF QUADRANTAL TRIANGLES.

154. A spherical triangle is called *quadrantal* when it has one side equal to a quadrant.

By Art. 136, (f), the polar triangle of a quadrantal triangle is a *right* triangle.

Therefore to solve a quadrantal triangle we have only to solve its polar triangle, and take the *supplements* of the parts obtained by the calculation.

1. Given $c = 90^\circ$, $a = 67^\circ\, 38'$, $b = 48^\circ\, 50'$; find A, B, and C.

Denoting the polar triangle by $A'B'C'$, we have by Art. 136, (f):

$C' = 90°$, $A' = 112° \, 22'$, $B' = 131° \, 10'$; to find a', b', and c'.

By Art. 144, the formulæ for the solution are

$$\overset{-}{\cos a'} = \frac{\overset{-}{\cos A'}}{\underset{+}{\sin B'}}, \quad \overset{-}{\cos b'} = \frac{\overset{-}{\cos B'}}{\underset{+}{\sin A'}}, \quad \overset{+}{\cos c'} = \overset{-}{\cot A'} \, \overset{-}{\cot B'}.$$

The check formula is $\quad \cos c' = \cos a' \cos b'$.

$$\begin{array}{ll}
\log \cos A' = 9.5804 & \log \cot A' = 9.6143 \\
\log \sin B' = \underline{9.8767} & \log \cot B' = \underline{9.9417} \\
\log \cos a' = 9.7037 & \log \cos c' = 9.5560 \\
\therefore \ 180° - a' = 59° \, 38.2'. & \therefore \ c' = 68° \, 54.8'.
\end{array}$$

$$\begin{array}{ll}
\log \cos B' = 9.8184 & \textit{Check.} \\
\log \sin A' = \underline{9.9660} & \log \cos a' = 9.7037 \\
\log \cos b' = 9.8524 & \log \cos b' = \underline{9.8524} \\
\therefore \ 180° - b' = 44° \, 36.7'. & \log \cos c' = 9.5561
\end{array}$$

Then in the given quadrantal triangle, we have

$$A = 180° - a' = \ \ 59° \, 38.2',$$

$$B = 180° - b' = \ \ 44° \, 36.7',$$

$$C = 180° - c' = 111° \ \ 5.2'.$$

EXAMPLES.

Solve the following quadrantal triangles:

2. Given $A = 139°$, $\quad b = 143°$.

3. Given $A = 45° \, 30'$, $B = 139° \, 20'$.

4. Given $a = 30° \, 20'$, $C = 42° \, 40'$.

5. Given $B = 70° \, 12'$, $C = 106° \, 25'$.

6. Given $A = 105° \, 53'$, $a = 104° \, 54'$.

XII. SPHERICAL OBLIQUE TRIANGLES.

GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

155. *In any spherical triangle, the sines of the sides are proportional to the sines of their opposite angles.*

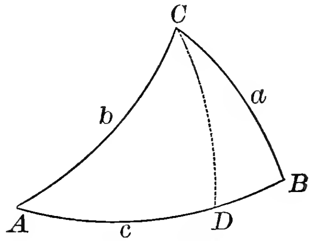


FIG. 1.

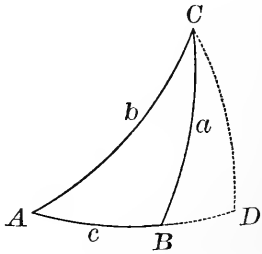


FIG. 2.

Let ABC be any spherical triangle, and draw the arc CD perpendicular to AB.

There will be two cases according as CD falls upon AB (Fig. 1), or upon AB produced (Fig. 2).

In the right triangle ACD, in either figure, we have by Art. 144,

$$\sin A = \frac{\sin CD}{\sin b}.$$

Also, in Fig. 1,

$$\sin B = \frac{\sin CD}{\sin a},$$

and in Fig. 2,

$$\begin{aligned} \sin B &= \sin(180° - CBD) \\ &= \sin CBD \quad \text{(Art. 47)} \\ &= \frac{\sin CD}{\sin a}. \end{aligned}$$

Dividing these equations, we have in either case

$$\frac{\sin A}{\sin B} = \frac{\dfrac{\sin CD}{\sin b}}{\dfrac{\sin CD}{\sin a}} = \frac{\sin a}{\sin b}. \tag{84}$$

In like manner we have,

$$\frac{\sin B}{\sin C}=\frac{\sin b}{\sin c}, \tag{85}$$

and
$$\frac{\sin C}{\sin A}=\frac{\sin c}{\sin a}. \tag{86}$$

The above results may be expressed more compactly as follows:

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

156. *In any spherical triangle, the cosine of either side is equal to the product of the cosines of the other two sides, plus the continued product of their sines and the cosine of their included angle.*

In the right triangle BCD, in Fig. 1 of the preceding article, we have by Art. 144,

$$\begin{aligned}\cos a &= \cos BD \cos CD\\ &= \cos (c-AD) \cos CD.\end{aligned}$$

And in Fig. 2,

$$\begin{aligned}\cos a &= \cos BD \cos CD\\ &= \cos (AD-c) \cos CD.\end{aligned}$$

Whence, in either case,

$$\cos a=\cos c \cos AD \cos CD+\sin c \sin AD \cos CD.$$

But in the right triangle ACD, by Art. 144,

$$\cos AD \cos CD=\cos b.$$

Also,

$$\begin{aligned}\sin AD \cos CD &= \sin AD \frac{\cos b}{\cos AD}=\cos b \tan AD\\ &= \sin b \frac{\tan AD}{\tan b}=\sin b \cos A \quad \text{(Art. 144)}.\end{aligned}$$

Whence,

$$\cos a=\cos b \cos c+\sin b \sin c \cos A. \tag{87}$$

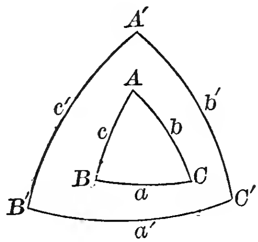
In like manner we have,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \qquad (88)$$

and

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \qquad (89)$$

157. Let ABC and $A'B'C'$ be a pair of polar triangles.



Applying the theorem of Art. 156 to the side a' of the triangle $A'B'C'$, we obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Putting for a', b', c', and A' their values as given in Art. 136, (f), we have

$$\cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a).$$

Whence by Art. 47,

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

That is,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \qquad (90)$$

In like manner,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \qquad (91)$$

and

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \qquad (92)$$

The above proof illustrates a very important application of the theory of polar triangles in Spherical Trigonometry.

If any relation has been found between the elements of a triangle, an analogous relation may be at once derived from it, in which each side or angle is replaced by the opposite angle or side, with suitable modifications in the algebraic signs.

158. *To express the sines, cosines, and tangents of the half-angles of a spherical triangle in terms of the sides of the triangle.*

From (**87**), Art. 156, we obtain

$$\sin b \sin c \cos A = \cos a - \cos b \cos c,$$

or, $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \qquad \textbf{(B)}$$

Subtracting both members from unity,

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}.$$

Whence by (**A**), Art. 75,

$$2 \sin^2 \tfrac{1}{2} A = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

But by Art. 72,

$$\cos y - \cos x = 2 \sin \tfrac{1}{2} (x + y) \sin \tfrac{1}{2} (x - y).$$

Whence,

$$2 \sin^2 \tfrac{1}{2} A = \frac{2 \sin \frac{1}{2} [a + (b - c)] \sin \frac{1}{2} [a - (b - c)]}{\sin b \sin c},$$

or, $$\sin^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c}.$$

Denoting $a + b + c$ by $2s$, so that s is the half-sum of the sides of the triangle, we have

$$a + b - c = (a + b + c) - 2c = 2s - 2c = 2(s - c),$$

and $$a - b + c = (a + b + c) - 2b = 2s - 2b = 2(s - b).$$

Whence,

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c},$$

and $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \qquad \textbf{(93)}$$

In like manner we have,

$$\sin \tfrac{1}{2} B = \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin c \sin a}}, \tag{94}$$

and

$$\sin \tfrac{1}{2} C = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}}. \tag{95}$$

Again, adding both members of (B) to unity, we have

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c},$$

or,

$$2 \cos^2 \tfrac{1}{2} A = \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

Whence, as in the proof of (93),

$$\cos^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(b + c + a) \sin \frac{1}{2}(b + c - a)}{\sin b \sin c}.$$

But, $b + c + a = 2s$, and $b + c - a = 2(s - a)$.

Whence,

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s-a)}{\sin b \sin c},$$

and

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}. \tag{96}$$

In like manner,

$$\cos \tfrac{1}{2} B = \sqrt{\frac{\sin s \sin (s-b)}{\sin c \sin a}}, \tag{97}$$

and

$$\cos \tfrac{1}{2} C = \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}. \tag{98}$$

Dividing (93) by (96), we obtain

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \sqrt{\frac{\sin b \sin c}{\sin s \sin (s-a)}}$$

$$= \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}. \tag{99}$$

In like manner,

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}}, \tag{100}$$

and

$$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}. \tag{101}$$

159. *To express the sines, cosines, and tangents of the half-sides of a spherical triangle in terms of the angles of the triangle.*

From (90), Art. 157, we obtain

$$\sin B \sin C \cos a = \cos A + \cos B \cos C,$$

or,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \tag{C}$$

Then, $1 - \cos a = 1 - \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$,

or,

$$2 \sin^2 \tfrac{1}{2} a = \frac{-(\cos B \cos C - \sin B \sin C) - \cos A}{\sin B \sin C}$$

$$= -\frac{\cos(B + C) + \cos A}{\sin B \sin C}.$$

Therefore by Art. 72,

$$2 \sin^2 \tfrac{1}{2} a = -\frac{2 \cos \frac{1}{2}(B+C+A) \cos \frac{1}{2}(B+C-A)}{\sin B \sin C}.$$

Denoting $A + B + C$ by $2S$, so that S is the half-sum of the angles of the triangle, we have $B + C - A = 2(S - A)$.

Whence, $\sin^2 \frac{1}{2} a = -\dfrac{\cos S \cos(S - A)}{\sin B \sin C}$,

and

$$\sin \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos(S - A)}{\sin B \sin C}}. \tag{102}$$

In like manner,

$$\sin \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos(S - B)}{\sin C \sin A}}, \tag{103}$$

and

$$\sin \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos(S - C)}{\sin A \sin B}}. \tag{104}$$

Again, adding both members of (**C**) to unity, we have

$$1+\cos a=1+\frac{\cos A+\cos B\cos C}{\sin B\sin C}$$

$$=\frac{\cos A+\cos B\cos C+\sin B\sin C}{\sin B\sin C}.$$

Then, $2\cos^2\frac{1}{2}a=\dfrac{\cos A+\cos(B-C)}{\sin B\sin C}$

$$=\frac{2\cos\frac{1}{2}[A+B-C]\cos\frac{1}{2}[A-(B-C)]}{\sin B\sin C},$$

or, $$\cos^2\tfrac{1}{2}a=\frac{\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A-B+C)}{\sin B\sin C}.$$

But $A+B-C=2(S-C)$, and $A-B+C=2(S-B)$.

Whence,

$$\cos^2\tfrac{1}{2}a=\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C},$$

and $$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}. \qquad \textbf{(105)}$$

In like manner,

$$\cos\tfrac{1}{2}b=\sqrt{\frac{\cos(S-C)\cos(S-A)}{\sin C\sin A}}, \qquad \textbf{(106)}$$

and $$\cos\tfrac{1}{2}c=\sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}}. \qquad \textbf{(107)}$$

Dividing (**102**) by (**105**), we obtain

$$\tan\tfrac{1}{2}a=\sqrt{-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}. \qquad \textbf{(108)}$$

In like manner,

$$\tan\tfrac{1}{2}b=\sqrt{-\frac{\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)}}, \qquad \textbf{(109)}$$

and $$\tan\tfrac{1}{2}c=\sqrt{-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}. \qquad \textbf{(110)}$$

NAPIER'S ANALOGIES.

160. Dividing (**99**), Art. 158, by (**100**), we have

$$\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \sqrt{\frac{\sin s \sin (s-b)}{\sin (s-c) \sin (s-a)}},$$

or, $$\frac{\sin \frac{1}{2} A \cos \frac{1}{2} B}{\cos \frac{1}{2} A \sin \frac{1}{2} B} = \sqrt{\frac{\sin^2 (s-b)}{\sin^2 (s-a)}} = \frac{\sin (s-b)}{\sin (s-a)}.$$

Whence, by composition and division,

$$\frac{\sin \frac{1}{2} A \cos \frac{1}{2} B + \cos \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} A \cos \frac{1}{2} B - \cos \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)}.$$

That is, by Arts. 65, 66, and 73,

$$\frac{\sin (\frac{1}{2} A + \frac{1}{2} B)}{\sin (\frac{1}{2} A - \frac{1}{2} B)} = \frac{\tan \frac{1}{2}[s-b+(s-a)]}{\tan \frac{1}{2}[s-b-(s-a)]}.$$

But $s-b+s-a=2s-a-b=c$.

Hence, $$\frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a-b)}. \qquad \textbf{(111)}$$

161. Multiplying (**99**) by (**100**), we have

$$\tan \tfrac{1}{2} A \tan \tfrac{1}{2} B = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}},$$

or, $$\frac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B} = \sqrt{\frac{\sin^2 (s-c)}{\sin^2 s}} = \frac{\sin (s-c)}{\sin s}.$$

Whence, by composition and division,

$$\frac{\cos \frac{1}{2} A \cos \frac{1}{2} B - \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\sin s - \sin (s-c)}{\sin s + \sin (s-c)},$$

or, by Art. 73, $$\frac{\cos (\frac{1}{2} A + \frac{1}{2} B)}{\cos (\frac{1}{2} A - \frac{1}{2} B)} = \frac{\tan \frac{1}{2}[s-(s-c)]}{\tan \frac{1}{2}[s+(s-c)]}.$$

But $s+s-c=2s-c=a+b$.

Hence, $$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a+b)}. \qquad \textbf{(112)}$$

162. Proceeding as in Art. 157, and applying (**111**) to the triangle $A'B'C'$, we obtain,

$$\frac{\sin\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'-b')}.$$

But, $\frac{1}{2}(A'+B')=\frac{1}{2}(180°-a+180°-b)$
$=180°-\frac{1}{2}(a+b),$

$\frac{1}{2}(A'-B')=\frac{1}{2}(180°-a-180°+b)=\frac{1}{2}(-a+b),$

$\frac{1}{2}c'=\frac{1}{2}(180°-C)=90°-\frac{1}{2}C,$

and $\frac{1}{2}(a'-b')=\frac{1}{2}(180°-A-180°+B)=\frac{1}{2}(-A+B).$

Whence,

$$\frac{\sin[180°-\frac{1}{2}(a+b)]}{\sin\frac{1}{2}(-a+b)}=\frac{\tan(90°-\frac{1}{2}C)}{\tan\frac{1}{2}(-A+B)}.$$

Therefore, by Arts. 42, 46, and 47,

$$\frac{\sin\frac{1}{2}(a+b)}{-\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{-\tan\frac{1}{2}(A-B)},$$

or, $$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}. \qquad (113)$$

In like manner, from (**112**), we obtain

$$\frac{\cos\frac{1}{2}(A'+B')}{\cos\frac{1}{2}(A'-B')}=\frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'+b')}.$$

But, $\frac{1}{2}(a'+b')=180°-\frac{1}{2}(A+B).$

Whence,

$$\frac{\cos[180°-\frac{1}{2}(a+b)]}{\cos\frac{1}{2}(-a+b)}=\frac{\tan(90°-\frac{1}{2}C)}{\tan[180°-\frac{1}{2}(A+B)]}.$$

Therefore, by Arts. 42, 46, and 47,

$$\frac{-\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{-\tan\frac{1}{2}(A+B)},$$

or, $$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}. \qquad (114)$$

163. The formulæ exemplified in Arts. 160, 161, and 162, are known as *Napier's Analogies.* In each case there may be other forms, according as other elements are used.

SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

164. In the solution of spherical oblique triangles, we may distinguish six cases:

1. *Given a side and two adjacent angles.*
2. *Given two sides and their included angle.*
3. *Given the three sides.*
4. *Given the three angles.*
5. *Given two sides and the angle opposite to one of them.*
6. *Given two angles and the side opposite to one of them.*

165. By application of the principles of Art. 136, (f), the solution of any example under Cases 2, 4, and 6 may be made to depend upon the solution of another example under Cases 1, 3, and 5, respectively; and *vice versa.*

Thus it is not essential to consider more than three cases in the solution of spherical oblique triangles.

166. The student must carefully bear in mind the remarks made in Arts. 151 and 152.

CASE I.

167. *Given a side and two adjacent angles.*

1. Given $A = 70°$, $B = 131° \, 20'$, $c = 116°$; find a, b, and C.

By Napier's Analogies (Arts. 160, 161), we have

$$\frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(b - a)},$$

and
$$\frac{\cos \frac{1}{2}(B + A)}{\cos \frac{1}{2}(B - A)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(b + a)}.$$

Whence,

$$\tan \tfrac{1}{2}(b-a) = \sin \tfrac{1}{2}(B-A) \csc \tfrac{1}{2}(B+A) \tan \tfrac{1}{2}c,$$

and $$\overset{-}{\tan} \tfrac{1}{2}(b+a) = \overset{+}{\cos} \tfrac{1}{2}(B-A) \overset{-}{\sec} \tfrac{1}{2}(B+A) \overset{+}{\tan} \tfrac{1}{2}c.$$

From the data,

$$\tfrac{1}{2}(B-A) = 30° \, 40', \quad \tfrac{1}{2}(B+A) = 100° \, 40', \quad \tfrac{1}{2}c = 58°.$$

$$\begin{array}{rlrl}
\log \sin \tfrac{1}{2}(B-A) &= 9.7076 & \log \cos \tfrac{1}{2}(B-A) &= 9.9346 \\
\log \csc \tfrac{1}{2}(B+A) &= 0.0076 & \log \sec \tfrac{1}{2}(B+A) &= 0.7326 \\
\log \tan \tfrac{1}{2}c &= \underline{0.2042} & \log \tan \tfrac{1}{2}c &= \underline{0.2042} \\
\log \tan \tfrac{1}{2}(b-a) &= 9.9194 & \log \tan \tfrac{1}{2}(b+a) &= 0.8714 \\
\therefore \tfrac{1}{2}(b-a) &= 39° \, 42.8'. & 180° - \tfrac{1}{2}(b+a) &= 82° \, 20.5' \\
 & & \therefore \tfrac{1}{2}(b+a) &= 97° \, 39.5'.
\end{array}$$

Then, $$a = \tfrac{1}{2}(b+a) - \tfrac{1}{2}(b-a) = 57° \, 56.7',$$

and $$b = \tfrac{1}{2}(b+a) + \tfrac{1}{2}(b-a) = 137° \, 22.3'.$$

To find C, we have by Art. 162,

$$\cot \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(b+a)}{\sin \frac{1}{2}(b-a)} \tan \tfrac{1}{2}(B-A)$$

$$= \sin \tfrac{1}{2}(b+a) \csc \tfrac{1}{2}(b-a) \tan \tfrac{1}{2}(B-A).$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(b+a) &= 9.9961 \\
\log \csc \tfrac{1}{2}(b-a) &= 0.1946 \\
\log \tan \tfrac{1}{2}(B-A) &= \underline{9.7730} \\
\log \cot \tfrac{1}{2}C &= 9.9637
\end{aligned}$$

$$\therefore \tfrac{1}{2}C = 47° \, 23.6', \text{ and } C = 94° \, 47.2'.$$

Note 1. The value of C may also be determined by the formula

$$\cot \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(b+a)}{\cos \frac{1}{2}(b-a)} \tan \tfrac{1}{2}(B+A) \quad \text{(Art. 162).}$$

Note 2. The triangle is always possible for any values of the given elements.

EXAMPLES.

Solve the following triangles:

2. Given $A = 78°$, $B = 41°$, $c = 108°$.

3. Given $B = 135°$, $C = 50°$, $a = 70° 20'$.

4. Given $A = 31° 40'$, $C = 122° 20'$, $b = 40° 40'$.

5. Given $A = 108° 12'$, $B = 145° 46'$, $c = 126° 32'$.

Case II.

168. *Given two sides and their included angle.*

1. Given $b = 137° 20'$, $c = 116°$, $A = 70°$; find B, C, and a.

By Napier's Analogies (Art. 162),

$$\frac{\sin\frac{1}{2}(b+c)}{\sin\frac{1}{2}(b-c)} = \frac{\cot\frac{1}{2}A}{\tan\frac{1}{2}(B-C)},$$

and
$$\frac{\cos\frac{1}{2}(b+c)}{\cos\frac{1}{2}(b-c)} = \frac{\cot\frac{1}{2}A}{\tan\frac{1}{2}(B+C)}.$$

Whence,

$$\tan\tfrac{1}{2}(B-C) = \sin\tfrac{1}{2}(b-c)\csc\tfrac{1}{2}(b+c)\cot\tfrac{1}{2}A,$$

and
$$\overset{-}{\tan\tfrac{1}{2}(B+C)} = \overset{+}{\cos\tfrac{1}{2}(b-c)}\;\overset{-}{\sec\tfrac{1}{2}(b+c)}\;\overset{+}{\cot\tfrac{1}{2}A}.$$

From the data,

$$\tfrac{1}{2}(b-c) = 10° 40',\quad \tfrac{1}{2}(b+c) = 126° 40',\quad \tfrac{1}{2}A = 35°.$$

From which we find,

$$\tfrac{1}{2}(B-C) = 18° 14.5',\quad \tfrac{1}{2}(B+C) = 113° 2.9'.$$

Then, $$B = \tfrac{1}{2}(B+C) + \tfrac{1}{2}(B-C) = 131° 17.4',$$

and $$C = \tfrac{1}{2}(B+C) - \tfrac{1}{2}(B-C) = 94° 48.4'.$$

To find a, we have by Art. 160,

$$\tan\tfrac{1}{2}a = \frac{\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}(B-C)}\tan\tfrac{1}{2}(b-c).$$

From which we obtain $a = 57° 56.6'$.

Note. The triangle is always possible for any values of the given elements.

EXAMPLES.

Solve the following triangles:

2. Given $a = 72°$, $b = 47°$, $C = 33°$.

3. Given $a = 98°$, $c = 60°$, $B = 110°$.

4. Given $b = 120° 20'$, $c = 70° 40'$, $A = 50°$.

5. Given $a = 125° 10'$, $b = 153° 50'$, $C = 140° 20'$.

Case III.

169. *Given the three sides.*

1. Given $a = 60°$, $b = 137° 20'$, $c = 116°$; find A, B, and C.

By Art. 158,

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}},$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}},$$

and
$$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}}.$$

From the data,

$$2s = a + b + c = 313° 20', \text{ or } s = 156° 40'.$$

Whence,

$$s - a = 96° 40', \; s - b = 19° 20', \; s - c = 40° 40'.$$

$$\begin{aligned} \log \sin (s-b) &= 9.5199 \\ \log \sin (s-c) &= 9.8140 \\ \log \csc s &= 0.4022 \\ \log \csc (s-a) &= 0.0029 \\ & 2\overline{)9.7390} \\ \log \tan \tfrac{1}{2} A &= 9.8695 \end{aligned}$$

$$\therefore \tfrac{1}{2} A = 36° 31.2', \text{ and } A = 73° 2.4'.$$

In like manner we find,

$$B = 131° \, 32.2', \quad \text{and} \quad C = 96° \, 55.4'.$$

The values of A, B, and C may also be obtained by aid of the sine or cosine formulæ of Art. 158.

If all the angles are to be computed, the tangent formulæ are the most convenient, as only four different logarithms are required. If but one angle is required, the cosine formulæ will be found to involve the least work.

Note. The triangle is always possible for any values of the given elements which satisfy the conditions of Art. 138, (a) and (c); that is, if $a + b + c$ is $< 360°$, and no side is greater than the sum of the other two.

EXAMPLES.

Solve the following triangles:

2. Given $a = 38°$, $b = 51°$, $c = 42°$.

3. Given $a = 101°$, $b = 49°$, $c = 60°$.

4. Given $a = 61°$, $b = 39°$, $c = 92°$.

5. Given $a = 62° \, 20'$, $b = 54° \, 10'$, $c = 97° \, 50'$.

Case IV.

170. *Given the three angles.*

1. Given $A = 70°$, $B = 131° \, 10'$, $C = 94° \, 50'$; find a, b, and c.

By Art. 159,

$$\tan \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}},$$

$$\tan \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos (S - B)}{\cos (S - C) \cos (S - A)}},$$

and $$\tan \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)}}.$$

From the data,

$$2S = A + B + C = 296°, \text{ or } S = 148°;$$

$$S - A = 78°, \ S - B = 16° \, 50', \ S - C = 53° \, 10'.$$

Note 1. Since $\cos S$ is $-$ (Art. 37), while the cosines of $S-A$, $S-B$, and $S-C$ are $+$, the quantities under the radical signs are essentially positive, and hence no attention need be paid to the negative signs in the formulæ.

$$\begin{aligned}
\log \cos S &= \ 9.9284 \\
\log \cos (S - A) &= \ 9.3179 \\
\log \sec (S - B) &= \ 0.0190 \\
\log \sec (S - C) &= \ 0.2222 \\
2 &\overline{)\,9.4875} \\
\log \tan \tfrac{1}{2} a &= \ 9.7437 \\
\therefore \ \tfrac{1}{2} a &= \ 28° \, 59.7', \text{ and } a = 57° \, 59.4'.
\end{aligned}$$

In like manner we find,

$$b = 137° \, 11.8', \quad \text{and} \quad c = 115° \, 55.8'.$$

The values of a, b, and c may also be obtained by aid of the sine or cosine formulæ of Art. 159.

If all the sides are to be computed, the tangent formulæ are the most convenient, as only four different logarithms are required. If but one angle is required, the sine formulæ will be found to involve the least work.

Note 2. The triangle is always possible for any values of the given elements, provided S is between 90° and 270°, and each of the quantities $S-A$, $S-B$, and $S-C$ between 90° and $-90°$ (Art. 37).

EXAMPLES.

Solve the following triangles:

2. Given $A = 75°$, $B = 82°$, $C = 61°$.

3. Given $A = 120°$, $B = 130°$, $C = 80°$.

4. Given $A = 91° \, 10'$, $B = 85° \, 40'$, $C = 72° \, 30'$.

5. Given $A = 138° \, 16'$, $B = 31° \, 11'$, $C = 35° \, 53'$.

Case V.

171. *Given two sides and the angle opposite to one of them.*

1. Given $a = 58°$, $b = 137° 20'$, $B = 131° 20'$; find A, C, and c.

By Art. 155, $\dfrac{\sin A}{\sin B} = \dfrac{\sin a}{\sin b}$,

or, $\sin A = \sin a \csc b \sin B$.

$$\begin{aligned} \log \sin a &= 9.9284 \\ \log \csc b &= 0.1689 \\ \log \sin B &= 9.8756 \\ \hline \log \sin A &= 9.9729 \end{aligned}$$

$\therefore A = 69° 58'$, or $110° 2'$ (Art. 152).

To find C and c, we have by Arts. 160 and 162,

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)} \tan \tfrac{1}{2}(B - A),$$

and
$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a).$$

Using the first value of A, we have

$$\tfrac{1}{2}(B + A) = 100° 39', \quad \tfrac{1}{2}(B - A) = 30° 41'.$$

Also, $\tfrac{1}{2}(b + a) = 97° 40'$, $\tfrac{1}{2}(b - a) = 39° 40'$.

From which we obtain

$$C = 94° 41.6', \text{ and } c = 115° 53.6'.$$

Using the second value of A, we have

$$\tfrac{1}{2}(B + A) = 120° 41', \quad \tfrac{1}{2}(B - A) = 10° 39'.$$

From which we obtain

$$C = 147° 26.4', \text{ and } c = 150° 56.8'.$$

Thus the two solutions are:

1. $A = 69° 58'$, $C = 94° 41.6'$, $c = 115° 53.6'$.
2. $A = 110° 2'$, $C = 147° 26.4'$, $c = 150° 56.8'$.

As in the corresponding case in the solution of plane oblique triangles (compare Arts. 124 to 126), there may sometimes be two solutions, sometimes only one, and sometimes none, in an example under Case V.

After the two values of A have been obtained, the number of solutions may be readily determined by inspection; for by Art. 136, (b), if a is $< b$, A must be $< B$, and if a is $> b$, A must be $> B$.

That is, *only those values of A can be retained which are greater or less than B according as a is greater or less than b.*

Thus, in Ex. 1, a is given $< b$; and since both values of A, $69° 58'$ and $110° 2'$, are $< B$, we have two solutions.

Also if the data are such as to make log sin A positive, there will be no solution corresponding.

2. Given $a = 58°$, $c = 116°$, $C = 94° 50'$; find A, B, and b.

In this case,
$$\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c},$$

or,
$$\sin A = \sin a \csc c \sin C.$$

$$\log \sin a = 9.9284$$
$$\log \csc c = 0.0463$$
$$\log \sin C = 9.9985$$
$$\log \sin A = 9.9732$$
$$\therefore A = 70° 5', \text{ or } 109° 55'.$$

Since a is given $< c$, only values of A which are $< C$ can be retained; hence there is but one solution, corresponding to the *acute* value of A.

To find B and b, we have by Arts. 160 and 162,

$$\cot \tfrac{1}{2} B = \frac{\sin \frac{1}{2}(c+a)}{\sin \frac{1}{2}(c-a)} \tan \tfrac{1}{2}(C-A),$$

and
$$\tan \tfrac{1}{2} b = \frac{\sin \frac{1}{2}(C+A)}{\sin \frac{1}{2}(C-A)} \tan \tfrac{1}{2}(c-a).$$

Using the first value of A, we have

$$\tfrac{1}{2}(C+A)=82^\circ\ 27.5', \quad \tfrac{1}{2}(C-A)=12^\circ\ 22.5';$$

also, $\quad \tfrac{1}{2}(c+a)=87^\circ, \qquad \tfrac{1}{2}(c-a)=29^\circ.$

From which we obtain

$$B=131^\circ\ 21.8', \text{ and } b=137^\circ\ 23.6'.$$

Thus the only solution is

$$A=70^\circ\ 5', \ B=131^\circ\ 21.8', \ b=137^\circ\ 23.6'.$$

3. Given $b=126^\circ$, $c=70^\circ$, $B=56^\circ$; find C.

In this case, $\quad \dfrac{\sin C}{\sin B}=\dfrac{\sin c}{\sin b},$

or, $\quad \sin C=\sin c \csc b \sin B.$

$$\begin{aligned} \log \sin c &= 9.9730 \\ \log \csc b &= 0.0920 \\ \log \sin B &= 9.9186 \\ \hline \log \sin C &= 9.9836 \end{aligned}$$

$$\therefore\ C=74^\circ\ 20', \text{ or } 105^\circ\ 40'.$$

Since both values of C are $>B$, while c is given $<b$, there is no solution.

EXAMPLES.

Solve the following triangles:

4. Given $b=99^\circ\ 40'$, $\quad c=64^\circ\ 20'$, $\quad B=95^\circ\ 40'$.

5. Given $a=40^\circ$, $\quad b=118^\circ\ 20'$, $\quad A=29^\circ\ 40'$.

6. Given $a=115^\circ\ 20'$, $\quad c=146^\circ\ 20'$, $\quad C=141^\circ\ 10'$.

7. Given $a=109^\circ\ 20'$, $\quad c=82^\circ$, $\quad A=107^\circ\ 40'$.

8. Given $b=108^\circ\ 30'$, $\quad c=40^\circ\ 50'$, $\quad C=39^\circ\ 50'$.

9. Given $a=162^\circ\ 20'$, $\quad b=15^\circ\ 40'$, $\quad B=125^\circ$.

10. Given $a=55^\circ$, $\quad c=138^\circ\ 10'$, $\quad A=42^\circ\ 30'$.

Case VI.

172. *Given two angles and the side opposite to one of them.*

1. Given $A = 110°$, $B = 131° 20'$, $b = 137° 20'$; find a, c, and C.

In this case, $$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B},$$

or, $$\sin a = \sin A \csc B \sin b.$$

$$\begin{aligned} \log \sin A &= 9.9730 \\ \log \csc B &= 0.1244 \\ \log \sin b &= 9.8311 \\ \hline \log \sin a &= 9.9285 \end{aligned}$$

$$\therefore a = 58° 1.2', \text{ or } 121° 58.8'.$$

To find c and C, we have by Arts. 160 and 162,

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a),$$

and $$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)} \tan \tfrac{1}{2}(B - A).$$

Using the first value of a, we have

$$c = 150° 53.8', \text{ and } C = 147° 23';$$

and using the second value of a,

$$c = 64° 7.8', \text{ and } C = 85° 17.6'.$$

Thus the two solutions are:

1. $a = 58° 1.2'$, $c = 150° 53.8'$, $C = 147° 23'$.
2. $a = 121° 58.8'$, $c = 64° 7.8'$, $C = 85° 17.6'$.

In Case VI., as well as in Case V., there are sometimes two solutions, sometimes only one, and sometimes none; and it may be shown, exactly as in Art. 171, that *only those values of a can be retained which are greater or less than b according as A is greater or less than B.*

Also if log sin a is positive, the triangle is impossible.

EXAMPLES.

Solve the following triangles:

2. Given $B = 116°$, $C = 80°$, $c = 84°$.

3. Given $A = 132°$, $B = 140°$, $b = 127°$.

4. Given $A = 62°$, $C = 102°$, $a = 64° 30'$.

5. Given $A = 133° 50'$, $B = 66° 30'$, $a = 81° 10'$.

6. Given $B = 22° 20'$, $C = 146° 40'$, $c = 138° 20'$.

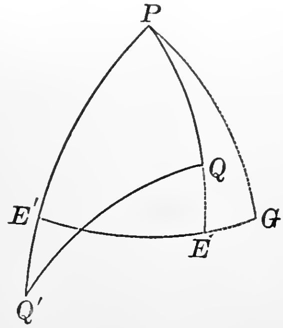
7. Given $A = 61° 40'$, $C = 140° 20'$, $c = 150° 20'$.

8. Given $B = 73°$, $C = 81° 20'$, $b = 122° 40'$.

APPLICATIONS.

173. In questions concerning geodesy or navigation, the earth may be regarded as a sphere.

The *shortest path* between any two points is the arc of a great circle which joins them, and the angles between this arc and the meridians of the points determine the bearings of the points from each other.



Thus, if Q and Q' are the points, and PQ and PQ' their meridians, the angle PQQ' determines the bearing of Q' from Q, and the angle $PQ'Q$ determines the bearing of Q from Q'.

If the latitudes and longitudes of Q and Q' are known, the arc QQ' and the angles PQQ' and $PQ'Q$ may be determined by the solution of a spherical triangle.

For if EE' is the equator, and PG the meridian of Greenwich, we have

$$\begin{aligned} \text{angle } QPQ' &= \text{angle } Q'PG - \text{angle } QPG \\ &= \text{longitude of } Q' - \text{longitude of } Q. \end{aligned}$$

Also, $\quad PQ = PE - QE = 90° - \text{latitude of } Q,$

and $\quad PQ' = PE' + Q'E' = 90° + \text{latitude of } Q'.$

Thus, in the triangle PQQ', two sides and their included angle are known, and the remaining elements may be computed.

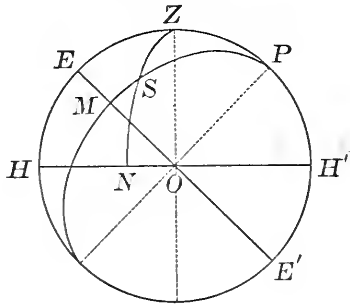
Note. When QQ' has been found in angular measure, its length in miles may be calculated by the method of Art. 132. In the following problems the diameter of the earth is taken as 7912 miles.

1. Boston lies in lat. 42° 21′ N., longitude 71° 4′ W.; and the latitude of Greenwich is 51° 29′ N. Find the shortest distance in miles between the places, and the bearing of each place from the other.

2. Calcutta lies in lat. 22° 33′ N., lon. 88° 19′ E.; and Valparaiso lies in lat. 33° 2′ S., lon. 71° 42′ W. Find the shortest distance in miles between the places, and the bearing of each place from the other.

3. Sandy Hook lies in lat. 40° 28′ N., lon. 74° 1′ W.; and Queenstown lies in lat. 51° 50′ N., lon. 8° 19′ W. In what latitude does a great circle course from Sandy Hook to Queenstown cross the meridian of 50° W.?

If the latitude of a place is known, and the altitude and declination of the sun, the solution of a spherical triangle serves to determine the hour of the day at the time and place of observation.



Thus let O be the position of the observer; P the celestial north pole; EE' the celestial equator; HH' the horizon; Z the zenith; S the sun's position; PSM a meridian passing through the sun's position; and ZSN a great circle passing through Z and S.

Then SM is the sun's declination, SN its altitude, and EZ the latitude of the place of observation.

Then in the spherical triangle SPZ, we have

$$SP = PM - SM = 90° - \text{the sun's declination},$$

$$SZ = ZN - SN = 90° - \text{the sun's altitude},$$

and $$PZ = EP - EZ = 90° - \text{the latitude of the place}.$$

That is, the three sides of the triangle SPZ are known, and the angle SPZ may be computed.

If 24 hours is multiplied by the ratio of this angle to 360°, we have the time required for the sun to move from S to the meridian EP.

Hence, if this time is subtracted from 12 o'clock, if the observation is made in the morning, or added, if made in the afternoon, we obtain the hour of the day at the time and place of observation.

If the Greenwich time of the observation is noted on a chronometer, the difference between this and the local time as calculated above serves to determine the longitude of the place of observation.

In reducing time to longitude, it should be borne in mind that 24 hours of time correspond to 360° of longitude; that is, one hour of time corresponds to 15° of longitude, one minute to 15′, and one second to 15″.

4. A mariner observes the altitude of the sun to be 14° 18′, its declination being 18° 36′ N. If the latitude of the vessel is 50° 13′ N., and the observation is made in the morning, find the hour of the day. If the observation is taken at 9 A.M., Greenwich time, what is the longitude of the vessel?

5. What will be the altitude of the sun at 4 P.M. in San Francisco, lat. 37° 48′ N., its declination being 12° S.?

6. In Melbourne, lat. 37° 49′ S., the altitude of the sun is observed to be 25° 46′. If the sun's declination is 3° S., and the observation is made in the morning, find the hour of the day.

7. At what hour will the sun rise in Boston, lat. 42° 21′ N., when its declination is 15° N.?

Note. At sunrise the sun's altitude is 0, so that the arc SZ becomes 90°.

FORMULÆ.

PLANE TRIGONOMETRY.

Art. 10.

$$\left.\begin{array}{lll} \sin A = \frac{a}{c}, & \tan A = \frac{a}{b}, & \sec A = \frac{c}{b}, \\ \cos A = \frac{b}{c}, & \cot A = \frac{b}{a}, & \csc A = \frac{c}{a}. \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{lll} \sin B = \frac{b}{c}, & \tan B = \frac{b}{a}, & \sec B = \frac{c}{a}, \\ \cos B = \frac{a}{c}, & \cot B = \frac{a}{b}, & \csc B = \frac{c}{b}. \end{array}\right\} \quad (2)$$

Art. 18.

$$\left.\begin{array}{lll} \sin A = \frac{1}{\csc A}, & \tan A = \frac{1}{\cot A}, & \sec A = \frac{1}{\cos A}, \\ \cos A = \frac{1}{\sec A}, & \cot A = \frac{1}{\tan A}, & \csc A = \frac{1}{\sin A}. \end{array}\right\} \quad (3)$$

Art. 19.

$$\sin^2 A + \cos^2 A = 1. \quad (4)$$

Art. 20.

$$\tan A = \frac{\sin A}{\cos A}. \quad (5) \qquad \cot A = \frac{\cos A}{\sin A}. \quad (6)$$

Art. 21.

$$\sec^2 A = 1 + \tan^2 A. \quad (7) \qquad \csc^2 A = 1 + \cot^2 A. \quad (8)$$

Art. 42.

$$\left.\begin{array}{ll} \sin(-A) = -\sin A, & \cos(-A) = \cos A, \\ \tan(-A) = -\tan A, & \cot(-A) = -\cot A, \\ \sec(-A) = \sec A, & \csc(-A) = -\csc A. \end{array}\right\} \quad (9)$$

Art. 44.

$$\left.\begin{array}{ll} \sin(90^\circ + A) = \cos A, & \cos(90^\circ + A) = -\sin A, \\ \tan(90^\circ + A) = -\cot A, & \cot(90^\circ + A) = -\tan A, \\ \sec(90^\circ + A) = -\csc A, & \csc(90^\circ + A) = \sec A. \end{array}\right\} \quad (10)$$

Art. 65. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$ (11)

$\cos(x+y) = \cos x \cos y - \sin x \sin y.$ (12)

Art. 66. $\sin(x-y) = \sin x \cos y - \cos x \sin y.$ (13)

$\cos(x-y) = \cos x \cos y + \sin x \sin y.$ (14)

Art. 71. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$ (15)

$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$ (16)

$\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}.$ (17)

$\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$ (18)

Art. 72. $\sin x + \sin y = 2 \sin \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y).$ (19)

$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y).$ (20)

$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y).$ (21)

$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x-y).$ (22)

Art. 73. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{1}{2}(x+y)}{\tan \frac{1}{2}(x-y)}.$ (23)

Art. 74. $\sin 2x = 2 \sin x \cos x.$ (24)

$\cos 2x = \cos^2 x - \sin^2 x.$ (25)

$\cos 2x = 1 - 2 \sin^2 x.$ (26)

$\cos 2x = 2 \cos^2 x - 1.$ (27)

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}.$ (28)

$\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}.$ (29)

Art. 75.

$$\sin\tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}. \quad (30) \qquad \cos\tfrac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}. \quad (31)$$

$$\tan\tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{1+\cos x}}. \quad (32)$$

$$\tan\tfrac{1}{2}x = \frac{\sin x}{1+\cos x}. \quad (33) \qquad \tan\tfrac{1}{2}x = \frac{1-\cos x}{\sin x}. \quad (34)$$

$$\cot\tfrac{1}{2}x = \frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}. \quad (35)$$

Art. 111.

$$4K = c^2 \sin 2A. \quad (36) \qquad 4K = c^2 \sin 2B. \quad (37)$$

$$2K = a^2 \cot A. \quad (38) \qquad 2K = b^2 \cot B. \quad (39)$$

$$2K = a^2 \tan B. \quad (40) \qquad 2K = b^2 \tan A. \quad (41)$$

$$2K = a\sqrt{(c+a)(c-a)}. \quad (42) \qquad 2K = b\sqrt{(c+b)(c-b)}. \quad (43)$$

$$2K = ab. \quad (44)$$

Art. 113.

$$\frac{\sin A}{\sin B} = \frac{a}{b}. \quad (45) \qquad \frac{\sin B}{\sin C} = \frac{b}{c}. \quad (46) \qquad \frac{\sin C}{\sin A} = \frac{c}{a}. \quad (47)$$

Art. 114.

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}. \quad (48)$$

$$\frac{b+c}{b-c} = \frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)}. \quad (49)$$

$$\frac{c+a}{c-a} = \frac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)}. \quad (50)$$

Art. 115.

$$\frac{a+b}{a-b} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}. \quad (51)$$

Art. 116.

$$a^2 = b^2 + c^2 - 2bc\cos A. \quad (52)$$

$$b^2 = c^2 + a^2 - 2ca\cos B. \quad (53)$$

$$c^2 = a^2 + b^2 - 2ab\cos C. \quad (54)$$

Art. 117. $$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{55}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca}. \tag{56}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}. \tag{57}$$

Art. 118. $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}. \tag{58}$$

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{ca}}. \tag{59}$$

$$\sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}. \tag{60}$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}. \tag{61}$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ca}}. \tag{62}$$

$$\cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}. \tag{63}$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \tag{64}$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}. \tag{65}$$

$$\tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \tag{66}$$

Art. 119.

$$2\,K = bc \sin A. \tag{67}$$

$$2\,K = ca \sin B. \tag{68}$$

$$2\,K = ab \sin C. \tag{69}$$

$$2\,K = \frac{a^2 \sin B \sin C}{\sin A}. \tag{70}$$

$$2\,K = \frac{b^2 \sin C \sin A}{\sin B}. \tag{71}$$

$$2\,K = \frac{c^2 \sin A \sin B}{\sin C}. \tag{72}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \tag{73}$$

SPHERICAL TRIGONOMETRY.

Art. 139.

$\cos c = \cos a \cos b.$ (74)

$\sin A = \dfrac{\sin a}{\sin c}.$ (75) $\sin B = \dfrac{\sin b}{\sin c}.$ (77)

$\cos A = \dfrac{\tan b}{\tan c}.$ (76) $\cos B = \dfrac{\tan a}{\tan c}.$ (78)

Art. 140.

$\tan A = \dfrac{\tan a}{\sin b}.$ (79) $\tan B = \dfrac{\tan b}{\sin a}.$ (80)

Art. 141.

$\sin A = \dfrac{\cos B}{\cos b}.$ (81) $\sin B = \dfrac{\cos A}{\cos a}.$ (82)

Art. 142.

$\cos c = \cot A \cot B.$ (83)

Art. 155. $\dfrac{\sin A}{\sin B} = \dfrac{\sin a}{\sin b}.$ (84)

$\dfrac{\sin B}{\sin C} = \dfrac{\sin b}{\sin c}.$ (85)

$\dfrac{\sin C}{\sin A} = \dfrac{\sin c}{\sin a}.$ (86)

Art. 156.

$\cos a = \cos b \cos c + \sin b \sin c \cos A.$ (87)

$\cos b = \cos c \cos a + \sin c \sin a \cos B.$ (88)

$\cos c = \cos a \cos b + \sin a \sin b \cos C.$ (89)

Art. 157.

$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$ (90)

$\cos B = -\cos C \cos A + \sin C \sin A \cos b.$ (91)

$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$ (92)

Art. 158. $$\sin\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}. \qquad (93)$$

$$\sin\tfrac{1}{2}B=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}}. \qquad (94)$$

$$\sin\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}. \qquad (95)$$

$$\cos\tfrac{1}{2}A=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}. \qquad (96)$$

$$\cos\tfrac{1}{2}B=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}}. \qquad (97)$$

$$\cos\tfrac{1}{2}C=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}. \qquad (98)$$

$$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}. \qquad (99)$$

$$\tan\tfrac{1}{2}B=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}. \qquad (100)$$

$$\tan\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}. \qquad (101)$$

Art. 159. $$\sin\tfrac{1}{2}a=\sqrt{-\frac{\cos S\cos(S-A)}{\sin B\sin C}}. \qquad (102)$$

$$\sin\tfrac{1}{2}b=\sqrt{-\frac{\cos S\cos(S-B)}{\sin C\sin A}}. \qquad (103)$$

$$\sin\tfrac{1}{2}c=\sqrt{-\frac{\cos S\cos(S-C)}{\sin A\sin B}}. \qquad (104)$$

$$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}. \qquad (105)$$

$$\cos\tfrac{1}{2}b=\sqrt{\frac{\cos(S-C)\cos(S-A)}{\sin C\sin A}}. \qquad (106)$$

$$\cos \tfrac{1}{2} c = \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}}. \qquad (107)$$

$$\tan \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}. \qquad (108)$$

$$\tan \tfrac{1}{2} b = \sqrt{-\frac{\cos S \cos (S - B)}{\cos (S - C) \cos (S - A)}}. \qquad (109)$$

$$\tan \tfrac{1}{2} c = \sqrt{-\frac{\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)}}. \qquad (110)$$

Art. 160.

$$\frac{\sin \frac{1}{2} (A + B)}{\sin \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a - b)}. \qquad (111)$$

Art. 161.

$$\frac{\cos \frac{1}{2} (A + B)}{\cos \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a + b)}. \qquad (112)$$

Art. 162.

$$\frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a - b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A - B)}. \qquad (113)$$

$$\frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A + B)}. \qquad (114)$$

ANSWERS.

Art. 9; page 3.

9. 28° 38′ 52.4″. **14.** 114° 35′ 29.6″.
13. 42° 58′ 18.6″. **15.** 100° 54′ 5.1″.
16. 30° 40′ 33.8″.

Art. 91; page 57.

2. .7781.	**7.** 1.3222.	**12.** 1.9912.	**17.** 2.1303.
3. 1.1461.	**8.** 1.7993.	**13.** 2.0212.	**18.** 2.2252.
4. .9030.	**9.** 1.7481.	**14.** 2.0491.	**19.** 2.1673.
5. 1.0791.	**10.** 1.9242.	**15.** 2.1582.	**20.** 2.5741.
6. 1.1761.	**11.** 1.6532.	**16.** 2.3343.	**21.** 2.5353.

Art. 93; page 58.

2. .3680.	**5.** 1.5441.	**8.** .2252.	**11.** .8539.
3. .1549.	**6.** .1182.	**9.** 2.2431.	**12.** .7660.
4. .5229.	**7.** 2.0970.	**10.** 1.0458.	**13.** .7360.

Art. 96; page 59.

3. .2863.	**9.** 4.5844.	**15.** .1165.	**22.** .2601.
4. 2.7090.	**10.** 3.2620.	**16.** .3860.	**23.** .6884.
5. 4.2255.	**11.** .9801.	**17.** .2212.	**24.** .1840.
6. .1398.	**12.** .4225.	**18.** .1750.	**25.** .2215.
7. .7194.	**13.** .1590.	**20.** 2.6145.	**26.** .2494.
8. .6611.	**14.** .0430.	**21.** .1678.	**27.** .1449.

Art. 98; page 60.

2. .2552.
3. .3522.
4. 9.2922 – 10.
5. 8.6811 – 10.
6. 1.5841.
7. 7.7323 – 10.
8. 6.4983 – 10.
9. 3.8663.
10. .6074.
11. 9.6511 – 10.
12. 2.4804.
13. 8.7905 – 10.
14. 6.3588.
15. .1964.
16. .1688.

Art. 105; pages 65 and 66.

1. 8.454.
2. 10.73.
3. – 2202.
4. .2179.
5. .01157.
6. – .7032.
7. 7.672.
8. .6688.
9. – 3.908.
10. 1782.
11. .3500.
12. – .4748.
13. .4127.
14. – 4.671.
15. .2415.
16. –.0725.
17. 13587.
18. .006415.
19. –1.184.
20. .000007038.
21. 2.924.
22. .9146.
23. 4.638.
24. .0000639.
25. 1.414.
26. 1.495.
27. – 1.246.
28. .6553.
29. .2846.
30. 2.372.
31. –.5142.
32. .1588.
35. 5.883.
36. .7885.
37. 1.195.
38. .6803.
39. .6443.
40. .5010.
41. 1.062.
42. – .9102.
43. 1.093.
44. .7035.
45. .5807.
46. – .6313.
47. 24.62.
48. .2979.
49. 98.50.
50. 1.660.
51. 3.076.
52. .8678.
53. 1.134.
54. .5881.
55. 1.805.
56. .003229.
57. .03344.

Art. 110; pages 69 to 72.

1. $a = 7.708$, $b = 8.124$.
2. $b = 1883$, $c = 2019.5$.
3. $a = 24.37$, $c = 53.56$.
4. $A = 43° \ 17.9'$, $b = .6622$.

5. $A = 68° 12.2'$, $c = 5.385$.

6. $b = 26.91$, $c = 87.64$.

7. $a = .02309$, $b = .01452$.

8. $A = 55° 36.1'$, $a = 4.216$.

9. $a = 5571$, $c = 7007$.

10. $A = 41° 2.4'$, $c = 153.8$.

11. $b = 167.6$, $c = 230.5$.

12. $a = 30.51$, $b = 18.59$.

13. $A = 47° 52.5'$, $b = 184.7$.

14. $a = 2.847$, $c = 3.287$.

15. $a = 5.4125$, $c = 14.306$.

16. $a = .7133$, $b = .1367$.

17. $A = 51° 51.9'$, $c = 811.7$.

18. $b = 42.27$, $c = 208.15$.

19. $A = 58° 35.7'$, $a = .0409$.

20. $b = 76.13$, $c = 1877$.

21. $A = 36° 45.9'$, $c = 41.22$.

22. $A = 65° 30'$, $a = 3.153$.

23. $a = 36992$, $b = 4021$.

24. $a = 410.5$, $c = 456.7$.

25. $A = 55° 44.1'$, $c = 411.5$.

26. $A = 58° 40'$, $b = 1.0405$.

27. $b = .3245$, $c = .8828$.

28. $a = 264.9$, $b = 75.95$.

29. $a = 176.64$, $c = 213.65$.

30. $A = 30° 17.2'$, $c = 20.04$.

31. $c = 25.206$.

32. $a = 1.735$.

33. $c = 2725.6$.

34. $A = 47° 42.9'$.

35. $a = .8346$.

36. $a = 49.25$.

37. 14.106 in.

38. 78.12 ft.

39. 31° 47.1′.

40. 36° 37.9′.

41. 99.45 miles.

42. 11.371 in.

43. 19° 50′.

44. 399.5 ft.

45. First, 25.22 miles; second, 30.07 miles.

46. 21.65 ft.

47. 14.487 in.; 15.682 in.

48. 453.7 ft.

49. 17.26 in.

50. 389.4 ft.

51. 437.6.

52. 5.773 in.

53. 481.9 ft.

54. Rate, 6.792 miles an hour; bearing, 63° 8.4′ west of north.

Art. 112; page 74.

2. 69.03.
3. .151.
4. 5695.
5. .08938.
6. 8208.
7. .002245.
8. 1.223.
9. 107.2.
10. .1017.

Art. 121; page 84.

2. $b = 15.837$, $c = 14.703$.
3. $a = .445$, $c = .9942$.
4. $a = .01913$, $b = .02272$.
5. $a = 33.78$, $c = 18.54$.
6. $b = 8.24$, $c = 5.464$.
7. $a = .011162$, $b = .006962$.
8. $b = 447.4$, $c = 425.7$.
9. $a = 342.6$, $c = 303.3$.
10. $a = 11.067$, $b = 6.067$.

Art. 122; page 85.

2. $A = 100° 56.7'$, $b = 19.78$.
3. $A = 83° 14.7'$, $c = 383.5$.
4. $B = 39° 15.8'$, $a = 3.211$.
5. $A = 24° 41.8'$, $c = .5886$.
6. $B = 143° 29.1'$, $a = 886.4$.
7. $A = 35° 1.9'$, $b = 12.78$.
8. $A = 59° 16.3'$, $c = 60.74$.
9. $B = 21° 1.2'$, $a = .06699$.
10. $A = 40° 52' 19''$, $b = 77.14$.

Art. 123; page 88.

3. $A = 28° 57.4'$, $B = 46° 34.2'$, $C = 104° 28.4'$.
4. $A = 34° 46.4'$, $B = 86° 24.8'$, $C = 58° 49'$.
5. $A = 74° 40'$, $B = 47° 46.4'$, $C = 57° 33.4'$.
6. $A = 58° 26'$, $B = 74° 23.8'$, $C = 47° 10.6'$.
7. $A = 55° 55.4'$, $B = 79° 43.8'$, $C = 44° 20'$.
8. $A = 49° 24.2'$, $B = 58° 38'$, $C = 71° 57.8'$.
9. $A = 60° 50.8'$, $B = 46° 6.2'$, $C = 73° 1.6'$.
10. $A = 18° 12.4'$, $B = 135° 50.6'$, $C = 25° 56.6'$.

Art. 127; pages 91 and 92.

6. $B = 39° 21.3'$, $c = 5.511$.
7. $B = 33° 28.4'$, $a = 118.33$;
or, $B = 146° 31.6'$, $a = 14.58$.
8. $C = 53° 18.9'$, $a = .07508$.
9. $A = 19° 18.1'$, $b = 2.522$.

10. $A = 79° 20'$, $c = .11416$;
or, $A = 100° 40'$, $c = .05121$.

11. Impossible.

12. $B = 24° 5.4'$, $a = 103.3$.

13. $C = 90°$, $b = 5.007$.

14. $C = 67° 10'$, $a = 6.918$;
or, $C = 112° 50'$, $a = 2.913$.

15. $A = 46° 53.3'$, $c = 141.48$.

16. Impossible.

17. $C = 24° 31.4'$, $b = 1.0637$.

18. $B = 90°$, $c = 127.32$.

19. $A = 70° 12'$, $b = .2879$;
or, $A = 109° 48'$, $b = .1045$.

20. $C = 37° 10'$, $a = 289.3$.

Art. 128; page 93.

2. 1077.9.
3. 14.697.
4. 3.257.
5. 5114.
6. 2604.
7. 1.353.
8. .06858.
9. 215.9.
10. .1273.
11. 4491.
12. .00001817.
13. 11.732.

Art. 129; pages 93 and 94.

1. Height of tower, 153.64 ft.; distance from first point, 117.27 ft.; from second, 217.27 ft.

2. 29800 square rods.

3. 247.7 ft.

4. 4.927 miles.

5. From first position, 9.9 miles; from second, 19.122 miles.

6. 298 ft.

7. Distance, 91.66 ft.; height, 33.9 ft.

8. 1113.1 ft.
9. 248 ft.
10. 1569.6.

Art. 153; page 112.

5.	$A = 148° 5'$,	$B = 65° 23.2'$,	$b = 38° 2'$.
6.	$a = 40° 41.8'$,	$b = 134° 30.8'$,	$c = 122° 7.5'$.
7.	$A = 149° 41'$,	$B = 66° 43.8'$,	$c = 137° 20'$.
8.	$A = 27° 11.6'$,	$a = 25° 25.2'$,	$c = 69° 54'$;
or,	$A = 152° 48.4'$,	$a = 154° 34.8'$,	$c = 110° 6'$.
9.	$A = 109° 22.5'$,	$a = 110° 57.6'$,	$b = 113° 22'$.
10.	$A = 66° 12.1'$,	$b = 127° 17.4'$,	$c = 107° 5.1'$.
11.	$A = 72° 28.9'$,	$B = 140° 38.1'$,	$c = 112° 37.7'$.
12.	$B = 27° 7.2'$,	$b = 25° 24.4'$,	$c = 109° 46'$;
or,	$B = 152° 52.8'$,	$b = 154° 35.6'$,	$c = 70° 14'$.
13.	$A = 120° 44.3'$,	$a = 156° 30'$,	$B = 33° 52.6'$.
14.	$a = 41° 5.5'$,	$b = 26° 25'$,	$c = 47° 32.1'$.
15.	$a = 60° 31.4'$,	$B = 143° 50'$,	$b = 147° 32.1'$.
16.	$A = 78° 46.7'$,	$a = 70° 10'$,	$c = 106° 27.5'$;
or,	$A = 101° 13.3'$,	$a = 109° 50'$,	$c = 73° 32.5'$.
17.	$A = 30° 32.1'$,	$a = 22° 1.1'$,	$b = 43° 17.9'$.
18.	$a = 166° 8.7'$,	$B = 101° 48.9'$,	$c = 50° 18.4'$.
19.	$A = 112° 2.5'$,	$B = 109° 12'$,	$c = 81° 53.6'$.
20.	$A = 135° 2.5'$,	$b = 68° 17.3'$,	$c = 105° 44'$.
21.	$a = 146° 32.5'$,	$b = 109° 48.1'$,	$c = 73° 35'$.
22.	$A = 70°$,	$B = 75° 6.2'$,	$b = 74° 7.3'$.

Art. 154; page 113.

2.	$a = 117° 1.2'$,	$B = 153° 41.9'$,	$C = 132° 34'$.
3.	$a = 57° 22.1'$,	$b = 129° 41.6'$,	$C = 57° 52.5'$.
4.	$A = 20° 0.9'$,	$B = 141° 29.6'$,	$b = 113° 17.1'$.
5.	$A = 33° 27.5'$,	$a = 35° 4.4'$,	$b = 78° 46.7'$.
6.	$B = 69° 16'$,	$b = 70°$,	$C = 84° 30'$;
or,	$B = 110° 44'$,	$b = 110°$,	$C = 95° 30'$.

Art. 167; page 125.

2.	$a = 95° 37.8'$,	$b = 41° 52.2'$,	$C = 110° 48.8'$.
3.	$b = 120° 16.6'$,	$c = 69° 19.6'$,	$A = 50° 26.2'$.
4.	$a = 34° 3'$,	$c = 64° 19.4'$,	$B = 37° 39.6'$.
5.	$a = 69° 4'$,	$b = 146° 25.6'$,	$C = 125° 12.2'$.

Art. 168; page 126.

2.	$A = 121° 32.8'$,	$B = 40° 56.8'$,	$c = 37° 25.8'$.
3.	$A = 86° 59.7'$,	$C = 60° 50.9'$,	$b = 111° 17'$.
4.	$B = 134° 57.3'$,	$C = 50° 41.1'$,	$a = 69° 8.8'$.
5.	$A = 147° 29'$,	$B = 163° 8.6'$,	$c = 76° 8.4'$.

Art. 169; page 127.

2.	$A = 51° 58.2'$,	$B = 83° 54.4'$,	$C = 58° 53.2'$.
3.	$A = 142° 32.8'$,	$B = 27° 52.6'$,	$C = 32° 27.2'$.
4.	$A = 35° 31'$,	$B = 24° 42.6'$,	$C = 138° 24.8'$.
5.	$A = 47° 21.2'$,	$B = 42° 19.4'$,	$C = 124° 38'$.

Art. 170; page 128.

2.	$a = 67° 51.8'$,	$b = 71° 44.4'$,	$c = 57°$.
3.	$a = 144° 9.8'$,	$b = 148° 48.6'$,	$c = 41° 43.6'$.
4.	$a = 89° 51.2'$,	$b = 85° 49.2'$,	$c = 72° 31.8'$.
5.	$a = 100° 4.6'$,	$b = 49° 58.8'$,	$c = 60° 6'$.

Art. 171; page 131.

4.	$C = 65° 30'$,	$A = 97° 20'$,	$a = 100° 44.6'$.
5.	$B = 42° 40'$,	$C = 159° 54'$,	$c = 153° 30.2'$;
or,	$B = 137° 20'$,	$C = 50° 20.6'$,	$c = 90° 9.6'$.
6.	Impossible.		
7.	$C = 90°$,	$B = 113° 36.9'$,	$b = 114° 51.9'$.

8. $B = 68° 18'$, $A = 132° 33.8'$, $a = 131° 15.8'$;
or, $B = 111° 42'$, $A = 77° 4.6'$, $a = 95° 50'$.
9. Impossible.
10. $C = 146° 37.9'$, $B = 55° 0.6'$, $b = 96° 34.4'$.

Art. 172; page 133.

2. $b = 114° 50'$, $A = 79° 20'$, $a = 82° 56'$.
3. $a = 67° 24'$, $C = 164° 6.4'$, $c = 160° 6.4'$;
or, $a = 112° 36'$, $C = 128° 20.6'$, $c = 103° 2.4'$.
4. $c = 90°$, $B = 63° 42.7'$, $b = 66° 26.2'$.
5. Impossible.
6. $b = 27° 22.1'$, $A = 47° 21.2'$, $a = 117° 9.2'$.
7. $a = 43° 3.1'$, $B = 89° 23.8'$, $b = 129° 8.4'$;
or, $a = 136° 56.9'$, $B = 26° 58.6'$, $b = 20° 35.8'$.
8. Impossible.

Art. 173; pages 134 and 136.

1. Distance, 3275 miles; bearing of Greenwich from Boston, N. 53° 7′ E.; of Boston from Greenwich, N. 71° 39.4′ W.

2. Distance, 11010 miles; bearing of Calcutta from Valparaiso, S. 64° 20.4′ E.; of Valparaiso from Calcutta, S. 54° 54.6′ W.

3. Latitude 49° 58.4′ N.

4. 6 h., 0 m., 40 s., A.M.; longitude 44° 50′ W.

5. 15° 0.8′.

6. 8 h., 2 m., 40 s., A.M.

7. 5 h., 3 m., 26.4 s., A.M.